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# Solving Optimal Control Problem Via Chebyshev Wavelet

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# ABSTRACT

Over the last four decades, optimal control problem are solved using direct and indirect methods. Direct methods are based on using polynomials to represent the optimal problem. Direct methods can be implemented using either discretization or parameterization. The proposed method in my thesis is considered as a direct method in which the optimal control problem is directly converted into a mathematical programming problem. A wavelet-based method is presented to solve the non-linear quadratic optimal control problem. The Chebyshev wavelets functions are used as the basis functions. The proposed method is also based on the iteration technique which replaces the nonlinear state equations by an equivalent sequence of linear time-varying state equations which is much easier to solve. Numerical examples are presented to show the effectiveness of the method, several optimal control problems were solved, and the simulation results show that the proposed method gives good and comparable results with some other methods.

## ملخص

**؟؟ حل مشكلة التحكم الأمثل باستخدام تحويل المويجات شيبيشيف؟؟**

خلال الأربع عقود الماضية،هناك طرق عديدة تعتمد على استخدام متعددة الحدود المتعامدة اقترحت لحل أنواع مختلفة من مسائل التحكم الأمثل. هذه الطرق تنقسم إلى طرق مباشرة وطرق غير مباشرة. الطريقة المباشرة ممكن أن تنجز باستخدام التقسيم والبارامتريزيشن، الطريقة المقترحة في هذه الرسالة تعتبر طريقة مباشرة.
في هذه الرسالة قدمت طريقة تعتمد على المويجات ( الويفلت ) لحل مسألة التحكم الأمثل للأنظمة الغير خطية. استخدمت دالة شيبيشيف كدالة أساسية في هذا العمل.واستخدمت كذلك التقنية التكرارية. قدمت أمثلة رقمية لإثبات تأثير هذه الطريقة . كذلك قدمنا مسائل متعددة ، وقد أثبتت الحلول أن هذه الطريقة تعطي نتائج أفضل أو مشابهة مع مقارنتها لطرق أخرى .

# DEDICATION

*To all my family members who have been a constant source of motivation, inspiration, and support.*

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# CHAPTER 1 INTRODUCTION

## 1.1. Thesis Motivation

The goal of an optimal controller is the determination of the control signal such that a specified performance index is optimized, while at the same time keeping the system equations, initial condition, and any other constraints are satisfied. Many different methods have been introduced to solve optimal control problem for a system with given state equations. Examples of optimal control applications include environment, engineering, economics etc.

The most popular method to solve the optimal control problem is the Riccati method for quadratic cost functions however this method results in a set of usually complicated differential equations [1]. In the last few decades orthogonal functions have been extensively used in obtaining an approximate solution of problems described by differential equations [2], which is based on converting the differential equations into an integral equation through integration. The state and/or control involved in the equation are approximated by finite terms of orthogonal series and using an operational matrix of integration to eliminate the integral operations. The form of the operational matrix of integration depends on the choice of the orthogonal functions like Walsh functions, block pulse functions, Laguerre series, Jacobi series, Fourier series, Bessel series, Taylor series, shifted Legendry, Chebyshev polynomials, Hermit polynomials and Wavelet functions [3].

As we know nonlinear optimal control problem does not has an analytical solution as linear case so this reason motivates many researchers to try to find a solution to this problem. In most cases, if not all, these solutions are numerical i.e. approximate or suboptimal solutions.

In general there are two methods or approaches that are used to solve optimal control problems: the indirect and direct methods.

Indirect methods are usually employed by converting the optimal control problem into a two-point boundary value problem TPBVP and solving this new problem which is easier than the original problem or finding a solution that satisfies the Hamilton- Jacobi-Bellman equation. The main advantage of indirect methods is that the resulted solutions produce existence and uniqueness of results, exact solutions when the TPBVP can be solved analytically, and error estimates when it is solved numerically [4], in the other hand it has some disadvantages as [2]:

- ❖ The solution of the Hamilton-Jacobi-Bellman equation of general nonlinear optimal control problem is very difficult.

- ❖ The lack of robustness.
- ❖ The user must have a deep knowledge of the mathematical and physical of the system model.

To avoid these drawbacks and others many researchers were proposed direct methods to solve optimal control problems. The direct method [2], is based on nonlinear programming (NLP) approaches that transcribe optimal control problems into NLP problems and apply existing NLP techniques to solve them. In most of practical applications, the control problems are described by strongly nonlinear differential equations hard to be solved by indirect methods. For those cases, direct methods can provide another choice to find the solutions.

In this method, the optimal solution is obtained by direct minimization of the performance index subject to the constraints. Direct methods classified into either discretization or parameterization of the state and/or the control variables.

In discretization, many discrete points (samples) of the state and/or control variable are required in order to produce accurate results, which make the system of large dimension.

Parameterization can be implemented by one of the three ways [4]:
• Control parameterization, in this way we approximate the control variables by a finite series of known functions with unknown parameters, then the state variables are obtained as a function of the unknown parameters by integrating the system state equation, but this process is computationally expensive.
• Control-state parameterization, in this way we approximate both state and control variables by a finite series of known functions with unknown parameters, the resulted system would ends up with large unknown parameters.
• State parameterization is the least used method compared with control parameterization and control-state parameterization. In state parameterization, only some state variables are directly approximated by a finite series of known functions with unknown parameters. The remaining state and control variables are obtained as a function of the unknown parameters directly from the state equation(s).Though, state parameterization is not used extensively in optimal control.

In this work we choose control-state parameterization to solve optimal control problem because there is no need as in control parameterization to integrate the system state equations and the state constraints can be handled directly.

## 1.2. Wavelet and Optimal Control Problem

Lately wavelets have found their way into many different fields of science and engineering. Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet [5].

Several numerical methods have been proposed in the last four decades to solve various classes of optimal control problems which are based on orthogonal polynomials [2-3], also wavelets approach was used in several papers to solve optimal control problems [6-7].

The wavelets are very effective in approximating functions with discontinuities or sharp changes because they are not supported on the whole interval $a \leq t < b$ as other orthogonal functions are, so this approach is very important [8].

Generally the wavelets can not be obtained in closed form. One of the wavelets that can be obtained in closed form are the Chebyshev wavelets and Legendre wavelet .

Wavelet analysis allows us to represent a function in terms of a set of basis functions, called wavelets, which are localized both in space and time. Here a continuous function $\psi$ , called the mother wavelet, is introduced.

## 1.3. What is Wavelets ?

Wavelets are mathematical functions that cut up data into different frequency components, and then study each component with a resolution matched to its scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. Wavelets were developed independently in the fields of mathematics, quantum physics, electrical engineering, and seismic geology. Interchanges between these fields during the last ten years have led to many new wavelet applications such as image compression, turbulence, human vision, radar, and earthquake prediction [9].

Wavelet analysis is a powerful mathematical tool, so it has been widely used in image digital processing, quantum field theory, numerical analysis and many other fields in recent years.

Wavelets possess several useful properties, such as orthogonality, compact support, exact representation of polynomials to a certain degree, and the ability to represent functions at different levels of resolution.

The first mention of wavelets appeared is from A. Haar 1909. One property of the Haar wavelet is that it has compact support, which means that it vanishes outside of a finite interval. Unfortunately, Haar wavelets are not continuously differentiable which somewhat limits their applications.

In 1985, Stephane Mallat gave wavelets an additional jump-start through his work in digital signal processing. He discovered some relationships between quadrature mirror filters, pyramid algorithms, and orthonormal wavelet bases. Inspired in part by these results, Y. Meyer constructed the first non-trivial wavelets. Unlike the Haar wavelets, the Meyer wavelets are continuously differentiable; however they do not have compact support. After that Ingrid Daubechies used Mallat's work to construct a set of wavelet orthonormal basis functions that are perhaps the most elegant, and have become the cornerstone of wavelet applications today [9].

## 1.4. Thesis Goals

- ❖ The first goal of this thesis is to apply control state parameterization to the optimal control problem.
- ❖ The second goal is that using Chebyshev wavelets to parameterize the state and control variables to solve linear and nonlinear optimal control problem.
- ❖ The third goal is to solve the optimal control problem directly by converting it into a quadratic programming problems. So an iteration technique developed by Banks[10-13] is used to replace the original nonlinear dynamic system by a sequence of linear time varying dynamic system, then we compared the results versus previous works.

## 1.5. Thesis Contribution

The contribution of this thesis can be summarized as

- ✓ Presents an effective method to solve linear quadratic optimal control problems time in-variant systems using control-state parameterization via Chebyshev wavelets.
- ✓ Presents an effective method to solve linear quadratic optimal control problems time-varying systems using control-state parameterization via Chebyshev wavelets .
- ✓ Introducing a new form of matrix of product for Chebyshev wavelet.
- ✓ Presents a new method for solving nonlinear quadratic optimal control problems using iteration technique and control-state parameterization via Chebyshev wavelets.

## 1.6. Thesis Organization

The remaining chapters of this thesis are organized as follows:

Chapter two reviews the optimal control problem in general and discusses some of the important previous works that are proposed to solve the optimal control problem. In this chapter, the computational techniques and methods used to solve optimal control problems are classified into direct and indirect methods.

Chapter three presents a numerical method for solving the linear quadratic optimal control problems with time in-varying systems. The concept of control state parameterization via Chebyshev wavelet are discussed in this chapter. In addition, some of the important properties of Chebyshev wavelet are reviewed. An explicit formula to approximate the quadratic performance index using Chebyshev wavelet is introduced, at the end of the chapter, computational results of a standard two examples ( one state and two states ) are introduced and the results are compared with some other methods.

Chapter four describes a method for solving the linear quadratic optimal control problems with time varying systems. An explicit formula to approximate the quadratic performance index using Chebyshev wavelet is introduced.
Chapter five presents the core of this work, where a computational method for solving the nonlinear quadratic optimal control problem is introduced. In this chapter, the concept of the iteration technique is presented. To verify the proposed method, a standard example is solved for the purpose of comparison with other methods.
Finally, Chapter six contains the important conclusions of this work and recommendations for future work.

# CHAPTER 2 OPTIMAL CONTROL PROBLEM

## 2.1. Introduction and Literature Review

The optimization of a dynamic system is usually aimed to find the optimal control $u^*(t)$ in minimizing or maximizing some performance indices under various constraints, keeping at the same time the system physical constraints unchanged. The performance index or cost function can be considered as the desired specifications of the system. We discuss some of the important previous works presented to solve the optimal control problem. Many textbooks [1],[14] and survey papers [15], that solved optimal control problem were published.

Several methods that use the orthogonal functions have been proposed to solve the time varying linear quadratic optimal control problem [16]. These methods are basically based on either converting the two point boundary value problem into a set of algebraic equations or on converting the dynamic optimal control problem into a quadratic programming problem.

A few works have appeared recently that employ the recently developed wavelets to approximate the optimization problem [6-8]. The use of wavelets is very effective in approximating signals with discontinuities or fast changing edges because of the localization property of wavelets.

Recently, Haar wavelets and Legendre wavelets have been used, Haar wavelet orthogonal functions and their integration matrices used to optimize dynamic systems and to solve lumped and distributed parameter systems were done by Chen and Hsiao [7],[17]. Jaddu [18], used Chebyshev wavelet to solve the linear quadratic optimal control problem with terminal constraints. The method is based on converting the optimal control problem into mathematical programming and he used the operational matrix of differentiation.

Razzaghi and Yousefi [19] defined functions which they call Legendre wavelets, however, these functions are scaling functions and not wavelets. Ghasemi and Kajani [20] presented a solution of time-varying delay systems by Chebyshev wavelets.
Babolian and Fattahzadeh [21] presented operational matrix of integration of Chebyshev wavelets basis and the product operational matrix. Here we will present a wavelet-based numerical method to solve a nonlinear optimal control problem. The method is based on using Chebyshev scaling functions to approximate the state and control variables. So, the optimal control problem is transformed into a quadratic programming problem.

### 2.1.1. Orthogonal Functions

Special attention has been given to applications of orthogonal functions, such as Walsh functions, block-pulse functions, Fourier series, Laguerre polynomials, Legendre polynomials, and Chebyshev polynomials. There are three classes of sets of orthogonal functions that are widely used. The first includes sets of piecewise constant basis functions (such as the Walsh functions, block pulse functions, etc.). The second consists of sets of orthogonal polynomials (such as Legendre polynomials and Chebyshev polynomials, etc.). The third is the widely used sets of sine–cosine functions in Fourier series [20].

The main characteristic of using orthogonal functions is that it reduces the problems to solving a system of linear algebraic equations, thus to simplify the problem as well.

### 2.1.2. Optimal Control Problem

We used the Chebyshev wavelets to present a computational method of the time varying linear optimal control problem and solved the nonlinear optimal control problem using iteration technique. The method is based on approximating the optimization problem by a quadratic programming problem.

Then we can classify the basic optimal control problem into three elements:

1- The system which be controlled: Mathematically, it is represented as a set of state equations which are a set of first order differential equations

$$\dot{x} = f(x(t), u(t), t) \quad , \quad t \in [t_0, t_f] \tag{2.1}$$

where $x \in R^n$ the state vector , $u \in R^m$ is the control vector. $f$ is assumed continuous differentiable function with respect to all its arguments.

2- A set of initial conditions which indicate the system state values at initial time

$$x(t_0) = x_0 \tag{2.2}$$

where $x_0 \in R^n$ represents a known initial condition vector.

3- Plant performance index (specifications): The desired specifications of the system that needs to be minimized (or maximized). Mathematically, the performance index is represented by a scalar function given by

$$J = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt \tag{2.3}$$

where $t_0$ and $t_f$ are the initial and final time; $h$ and $g$ are scalar functions. $t_f$ may be specified or free, depending on the problem statement.

## 2.2. Problem Statement

We can state the general unconstrained optimal control problem as follows:
Find an optimal controller, feedback $u(x(t),t)$ if possible, or if not an open loop $u(t)$ that minimizes the following performance index

$$J = h\big(x(t_f),t_f\big) + \int_{t_0}^{t_f} g(x(t),u(t),t)dt \qquad (2.4)$$

Subject to

$$\dot{x} = f(x(t),u(t),t) \quad , \qquad x(t_0) = x_0 \qquad (2.5)$$

In general the previous problem ( 2.4 ), ( 2.5 ) can be solved by many methods. This problem basically can be solved by one of the following approaches [2]:

- ✓ Bellman's dynamic programming method (Hamilton-Jacobi-Bellman HJB Equation).
- ✓ Variational method and Pontryagin's minimum principle (Euler-Lagrange Equations).
- ✓ Direct methods using Parameterization or discretization (nonlinear mathematical programming).

Bellman's dynamic programming method is based on methods that satisfy HJB equation. The optimal controller resulted from these methods is a closed loop or feedback controller $u(x(t))$. Methods that are based on the variational method and Pontryagin's minimum principle (Euler-Lagrange equations) convert the optimal control problem into a Two-Point Bounded Value Problem (TPBVP). The optimal controller resulted from using these methods would also produce a feedback or closed loop controller $u(t)$. Methods that are based on HJB equation or Euler-Lagrange equations are usually classified as indirect methods.

Methods that are based on parameterization or discretization are called direct methods. Direct methods usually produce an open loop optimal controller $u(t)$. Direct methods are based on solving the optimal control problem by converting it into a nonlinear programming problem. The proposed method in this work is classified as a direct method, so we will discuss these two methods in the following sections .

## 2.3. Indirect Methods

An indirect method transforms the problem into another form before solving it. The indirect method is sometimes described as "*first optimize then discretize*." because optimality conditions are found before numerical techniques are introduced. As mentioned earlier, indirect methods are based on solutions that satisfy the HJB equation or on solutions that convert the optimal control problem into a TPBVP, we review this method as follows [2],

1. **Power series approach**, this approach is based on finding an approximate solution to the Hamilton-Jacobi-Bellman equation or the nonlinear two-point boundary value problem by using power series expansion. The approximated feedback control law obtained by this technique is solved successively.

The pioneers of this method are:

- ✓ Lukes [23] applied this method to obtain an approximated feedback control law of the HJB equation. Lukes assumed a general nonlinear infinite horizon (regulator) optimal control problem.

- ✓ Willemstein [24] extended the work of Lukes to handle finite time optimal control problems both fixed end and free end. The optimal control problem reduced to solving successively systems of ordinary differential equations.

- ✓ Garrard and Jordan [25] applied the work done by Lukes to control a complex dynamic system of an F8 fighter jet.

- ✓ Yoshida and Loparo [26] apply a similar idea of Lukes to solve a nonlinear optimal control problem with quadratic performance index for both finite and infinite time problems.

2.**Extended linearization method** [27] in this method, the nonlinear dynamic system expressed as a nonlinear state equations of the form

$$\dot{x} = f(x, u, t) \tag{2.6}$$

where $f(x, u, t)$ is a nonlinear function in $x$ is to be rewritten in a "pseudo" linear form

$$\dot{x} = Ax(t) + Bu(t) \tag{2.7}$$

3. **Inverse optimal control problem** [28] an optimal feedback control is obtained by finding a solution to the inverse optimal control problem.

## 2.4. Direct Methods

Direct methods are an important class of methods for solving the optimal control problem. Direct methods are employed by direct substitution of the state and control variables into the performance index without constructing the Hamiltonian of the system as in indirect methods, the direct method has been described as "*first discretize then optimize*."

These methods offer some advantages when applied to optimal control problems. The first advantage is that we can convert the difficult dynamic optimal control problem into static parameters optimization problem which is easier than the old one; many software packages are available to solve this static problem; we can deal with different constraints directly.

So, many techniques and methods were proposed that are based on direct methods. As a result the difficult nonlinear dynamic optimal control problem is converted into a nonlinear mathematical programming problem by using a direct method. Direct methods can be implemented by either using discretization or parameterization methods. Here, we will use parameterization technique to convert the difficult nonlinear quadratic optimal control problem into linear time-varying quadratic control problems which are much easier than the original problem. In the following sections, we will briefly discuss both discretization and parameterization technique and we will concentrate on parameterization.

### 2.4.1. Discretization:

Discretization is a process in which the time interval $t \in [t_0, t_f]$ is to be divided into an equal $n$ time segments, mathematically, this can be given as [29]

$$t_0 < t_1 < t_2 < t_3 < \cdots < t_n = t_f \qquad (2.8)$$

As a result, and depending on the discretization technique, the variable(s) is (are) sampled at each time point in ( 2.8 ). Basically, there are two discretization technique used in optimal control problem: Control-state discretization and control discretization.

**1. Control-State Discretization:**
Apply this method to discretize both state and control variables. As a result, the following vector which contains a sequence of unknown state and control variables will be produced [30]

$$y = (x_0, x_1, \dots \ , x_n, u_0, u_1, \dots \ u_{n-1}) \qquad (2.9)$$

By this, the system state equations are replaced by algebraic equations which are treated as equality constraints. This would convert the original optimal control problem into a static optimization problem that can be solved using any available software packages like MATLAB. Note that in order to have accurate results, large amount of samples should be taken, this would result in a system that is highly dimensional.

**2. Control Discretization:**
In this approach, is to discretize the control variables only. As a result, the following vector is obtained [30]

$$y = (u_0, u_1, \dots \ u_{n-1}) \qquad (2.10)$$

In order to get the state variables, it is necessary to integrate the system state equations.

This would produce state variables that are a function of the control variables. An advantage of this method over control-state discretization is that the resulted system is lower in dimension.

### 2.4.2. Parameterization

The Parameterization technique is an essential part of my thesis, it is a process in which a function or a variable is approximated using known functions with known or unknown parameters. Parameterization can be employed by one of the three forms: Control parameterization, control-state parameterization and state parameterization. In this work we will use control-state parameterization.

**1. Control Parameterization**

In this method, only the control variables are approximated by a finite length series of known functions with unknown parameters, mathematically, this can be formulated as follows

$$u_k = \sum_{i=0}^{N} b_i^{(k)} f_i\,(t) \qquad k = 1,2,\ldots\ldots,m \tag{2.11}$$

Where $N$ is the order of approximation, $b_i' s$ are the unknown parameters and $f_i' s$ are a suitably selected set of functions forming a basis of the control space. By integrating the state equation, the state variables can be obtained as a function of the unknown parameters of the control variables. Both control and state variables are then directly substituted into the performance index. By this, the original difficult optimal control problem is converted into a static optimization problem of the unknown parameters which can be solved using any available software packages as MATLAB. This method is the most widely used method compared to the other parameterization techniques. But, integration of the state equations to get the state variables is an expensive computation process [4].

**2. Control-State Parameterization**

Using this method, both control and state variables are approximated by a finite length series of known functions with unknown parameters of its own. Mathematically, this can formulated as follows [31]

$$x_k = \sum_{i=0}^{N} a_i^{(k)} f_i\,(t) \qquad k = 1,2,\ldots\ldots,n \tag{2.12}$$

$$u_l = \sum_{i=0}^{N} b_i^{(l)} f_i\,(t) \qquad l = 1,2,\ldots\ldots,m \tag{2.13}$$

Where $a_i' s$ , $b_i' s$ are the unknown parameters, $N$ is the order of approximation and $f_i$ is a suitably selected set of functions forming a basis. By this, the optimal control problem is converted into a nonlinear mathematical programming problem. Since both state and control variables are parameterized, the resulted system would ends up with a large number of unknown parameters.

**3. State Parameterization**

In this method, only the state variables are to be approximated by a finite length series of known functions with unknown parameters, mathematically, this can be formulated as follows [32]

$$x_k = \sum_{i=0}^{N} a_i^{(k)} f_i\,(t) \qquad\qquad k = 1,2,\ldots\ldots,n \qquad (2.14)$$

The control variables can be obtained from the state equations. The idea is to choose a set of state variables that are to be approximated directly by a finite length series of known functions with unknown parameters. The remaining state and control variables can be obtained as a function of the directly approximated state variables parameters from the system state equations. This would decrease the resulted system dimension dramatically. If any state equation remains unsatisfied, it will be considered as an equality constraint.

## 2.5. Advantages of Direct Methods over Indirect Methods

Using indirect method has certain advantages, which include existence and uniqueness of results, exact solutions when the TPBVP can be solved analytically, and error estimates when it is solved numerically [4]. There are many disadvantages of indirect method which can be overcome by a direct method. The first disadvantage of the indirect method is that each solution is problem specific; a separate set of mathematical transformation must be applied for each distinct optimal control problem. On the other hand direct method gives more universal solution; it is a numerical technique for solving a set of problems and can be very easily and quickly applied to the new set of equation without taking care of complication of problem. Second, in an indirect method, the transformation requires that the optimal control problem should be formulated with a single objective functional. When there are multiple objectives, they must be collected into one. In the direct method multiobjective global optimizer can be used to solve this type of problems. One numerical run can produce a range of solutions that can be considered mutually optimal in some sense. This provides a mathematical, rather than experimental, basis for generating a range of results from which to choose [33].

As a result, the range of problems that can be solved via direct methods is significantly larger than the range of problems that can be solved via indirect methods.

In general we can illustrate the computational methods of optimal control problem in block diagram as in Figure ( 2.1 ).

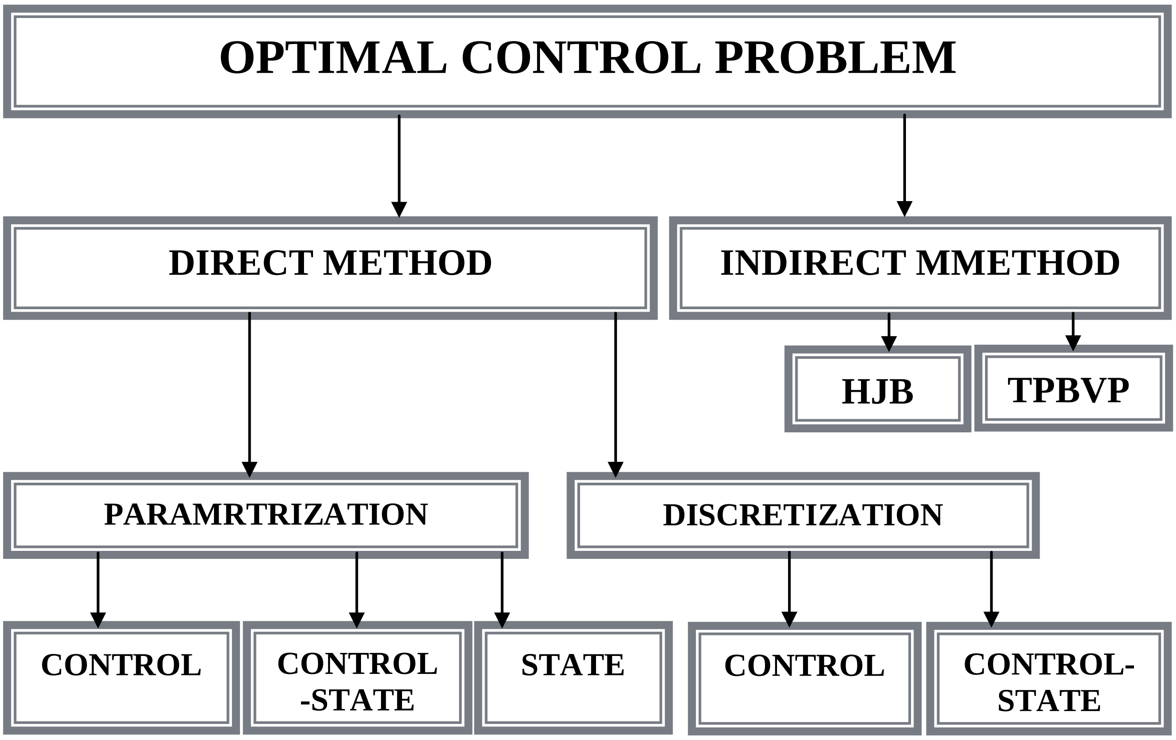


**Figure ( 2.1 )** Computation methods of optimal control problem

# CHAPTER 3 LINEAR TIME-INVARIANT QUADRATIC OPTIMAL CONTROL PROBLEM

## 3.1. Introduction

The main idea proposed in this work is to solve nonlinear optimal control problems by replacing the original nonlinear state equations by an equivalent sequence of linear time-varying state equations as we will seen in the iteration technique [10-13] in chapter five, so first we must study linear optimal control problems.

The linear optimal control problem is one of the few optimal control problems that can be solved analytically. The solution of this problem gives a feedback control law. This solution can be found in many text books like [1]. However, this solution is not that easy. In order to solve this problem, it is necessary to solve either the nonlinear matrix Riccati equation or to convert the problem into Two-Point Boundary Value Problem (TPBVP).

Many numerical methods which are based on orthogonal polynomials have been proposed to solve various classes of optimal control problems [34]. Lately several papers have appeared that are based on using wavelets approach to solve optimal control problems [17–20]. The reason for using wavelets approach is that the wavelets are very effective in approximating functions with discontinuities or sharp changes because they are not supported on the whole interval $a < t < b$ as other orthogonal functions are.

Here we used the direct method to solve the optimal control problem, some researchers proposed direct methods by using either discretization or parameterization to solve linear optimal control problems to avoid difficulties associated with solving using indirect methods, Razzaghi and Elnagar [16] parameterize the derivative of the state variables using shifted Legendre polynomials. Jaddu [2] proposed a method that is based on state parameterization using Chebyshev polynomials. Chen and Hsiao [7],[17] proposed a method using Haar wavelet orthogonal functions and their integration matrices to solve lumped and distributed parameter systems. Jaddu [18] also, used Chebyshev wavelet to solve the linear quadratic optimal control problem with terminal constraints. The method is based on converting the optimal control problem into mathematical programming and he used the operational matrix of differentiation. Babolian and Fattahzadeh [21] presented operational matrix of integration of Chebyshev wavelets basis and the product operational matrix.

In this chapter, we will propose a method to solve linear time-invariant quadratic optimal control problems using control state parameterization via Chebyshev wavelets by using operational matrix of integration.

The method is based on using Chebyshev scaling functions to approximate the state and control variables. So, the optimal control problem is transformed into a quadratic programming problem and solved it using MATLAB program.

## 3.2. Statement of Linear Quadratic Optimal Control Problem

We can stated the linear quadratic optimal control problem as follows:
Find an optimal controller $\overset{*}{u}(t)$ that minimizes the following quadratic performance index

$$J = \int_0^{t_f} (x^T Q x + u^T R u) dt \tag{3.1}$$

subject to the following linear dynamic system and initial conditions

$$\dot{x} = Ax(t) + Bu(t) \qquad\qquad x(0) = x_0 \tag{3.2}$$

where $x \in R^n$, $u \in R^m$, $x_o \in R^n$, $A, B$ are $n\,x\,n$ and $n\,x\,m$ real-valued matrices respectively. $Q$ is an $n\,x\,n$ positive semidefinite matrix and $R$ is an $m\,x\,m$ positive definite matrix, we will assume that $m \leq n\ and\ t \in [0, t_f]$.

The method proposed to solve the problem ( 3.1 ) , ( 3.2 ) is based on directly parameterizing the state and control variables by a finite length series of Chebyshev wavelets with unknown parameters.

## 3.3. Control State Parameterization via Chebyshev Wavelets

In this section, we will present the proposed method of solving optimal control problem by using control state parameterization via Chebyshev wavelets, before that we will review some of the important properties of Chebyshev wavelets.

### 3.3.1. Some Properties of Chebyshev Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter $a$ and the translation parameter $b$ vary continuously we have the following family of continuous wavelets as [21]

$$\Psi_{a,b}(t) = |a|^{-\frac{1}{2}} \Psi\left(\frac{t-b}{a}\right), \qquad a, b \in \mathbb{R}, a \neq 0 \tag{3.3}$$

Chebyshev wavelets $\psi_{nm}(t) = \psi(k, m, n, t)$ have four arguments; $k = 1,2,3, \dots,$ $n = 1,2,3, \dots, 2^k$, $m$ is the order for Chebyshev polynomials and t is the normalized time. They are defined on the interval [0,1) by:

$$\Psi_{nm}(t) = \begin{cases} \frac{\alpha_m 2^{\frac{k}{2}}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \frac{n-1}{2^k} \leq t \leq \frac{n}{2^k} \\ 0 & elsewhere \end{cases} \quad (3.4)$$

*where*

$$\alpha_m = \begin{cases} \sqrt{2} & m = 0 \\ 2\ , & m = 1,2,\ldots \end{cases}$$

Here, $T_m(t)$ are the well−known Chebyshev polynomials of order $m$, which are orthogonal with respect to the weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$ and satisfy the following recursive formula [20]:

$$T_0(t) = 1,$$
$$T_1(t) = t,$$
$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1,2,3,\ldots. \quad (3.5)$$

The set of Chebyshev wavelets are an orthogonal set with respect to the weight function

$$\omega_n(t) = \omega(2^{k+1}\, t - 2n + 1) \quad (3.6)$$

### 3.3.2. Function Approximation

A function $f(t)$ defined over [0,1) may be expanded as:

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{nm}\psi_{nm}(t) \quad (3.7)$$

*where*:

$$f_{nm} = (f(t), \psi_{nm}(t))$$

If the infinite series in Eq. ( 3.7 ) is truncated, then Eq. ( 3.7 ) can be written as:

$$f(t) \cong f_{2^k,M-1} = \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(t) = F^T\Psi(t) \quad (3.8)$$

$where\ F\ and\ \psi(t) are\ 2^k M\ x\ 1\ matrices\ given\ by:$

$$F = [f_{10}, f_{11}, \ldots., f_{1,M-1}, f_{20}, \ldots, f_{2,M-1}, \ldots f_{2^k,0}, \ldots, f_{2^k,M-1}]^T \quad (3.9)$$

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \ldots., \psi_{1,M-1}(t), \psi_{20}(t), \ldots, \psi_{2,M-1}(t), \ldots \psi_{2^k,0}(t), \ldots, \psi_{2^k,M-1}]^T \quad (3.10)$$

### 3.3.3. Chebyshev Wavelets Operational Matrix of Integration:

The power of orthogonal functions to construct operational matrices for solving identification and optimization problems of dynamics systems was start in 1975 when Chen and Hsiao initially established the Walsh-type operational matrix. Since then many operational matrices based on various orthogonal functions such as block pulse, Laguerre, Legendre, Chebyshev, as will as Fourier, have been developed . The main characteristic of this technique is to convert a differential equation into an algebraic one, and therefore the solution, and optimizing identification procedures are either reduced or simplified [17].

For Chebyshev wavelet the integration of the vector $\Psi(t)$ defined in Eq. ( 3.10 ) can be obtained as

$$\int_0^t \Psi(s)ds \cong P\Psi(t) \tag{3.11}$$

where P is the $(2^k M)\ x\ (2^k M)$ operational matrix for integration and is given in [20] as

$$P = \begin{bmatrix} C & S & S & \cdots & S \\ 0 & C & S & \cdots & S \\ 0 & 0 & C & \cdots & S \\ \vdots & \vdots & \vdots & \ddots & S \\ 0 & 0 & 0 & \cdots & C \end{bmatrix} \tag{3.12}$$

*Where C and S are M x M matrices given by* :

$$C = \frac{1}{2^k}\begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4\sqrt{2}} & 0 & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3\sqrt{2}} & \frac{-1}{4} & 0 & \frac{1}{12} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \cdots & \frac{-1}{4(M-3)} & 0 & \frac{-1}{4(M-1)} \\ \frac{-1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{4(M-2)} & 0 \end{bmatrix} \tag{3.13}$$

*and*

$$S = \frac{\sqrt{2}}{2^k}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{15} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{M(M-2)} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{3.14}$$

**Lemma 1**

The integration of the product of two Chebyshev wavelet function vectors is obtained as $for \quad k = 1,2,\ldots$ and $\mathrm{M} = 3$

$$\int_0^1 \Psi(t)\Psi^{\mathrm{T}}(t)dt = RR \qquad (3.15)$$

$where$
$$RR = \begin{bmatrix} G & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G \end{bmatrix}$$

$and$
$$G = \begin{bmatrix} \frac{2}{\pi} & 0 & -\frac{2\sqrt{2}}{3\pi} \\ 0 & \frac{4}{3\pi} & 0 \\ -\frac{2\sqrt{2}}{3\pi} & 0 & 0.594 \end{bmatrix}$$

The following property of the product of two Chebyshev wavelets vectors [20] will also be used. Let

$$\Psi(t)\Psi^{T}(t)F = \breve{F}\Psi(t), \qquad (3.16)$$

Where $\tilde{F}\ is\ (2^k M)x\ \ matrix. To\ illustrate\ the\ calculation\ procedure\ we\ choose$ $M = 3\ and\ k = 2.$

$Thus\ we\ have$:

$F = [f_{10}, f_{11}, f_{12}, \ldots, f_{40}, f_{41}, f_{42}]^T$

$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), C, \ldots, \psi_{4,0}(t), \psi_{41}(t), \psi_{42}(t)]^T$

### 3.3.4. Chebyshev Scaling Functions

$From\ Eq.$ ( 3.4 ) $we\ can\ obtained$ ( $when\ M = 3, k = 2$)

$$\left.\begin{aligned} \psi_{10}(t) &= \sqrt{\frac{8}{\pi}} \\ \psi_{10}(t) &= \frac{4}{\sqrt{\pi}}(8t-1) \\ \psi_{11}(t) &= \frac{4}{\sqrt{\pi}}(2(8t-1)^2-1) \end{aligned}\right\} 0 \le t \le \frac{1}{4} \qquad (3.17)$$

$$\left.\begin{aligned}\psi_{20}(t) &= \sqrt{\frac{8}{\pi}} \\ \psi_{21}(t) &= \frac{4}{\sqrt{\pi}}(8t-3) \\ \psi_{22}(t) &= \frac{4}{\sqrt{\pi}}(2(8t-3)^2-1)\end{aligned}\right\} \frac{1}{4} \leq t \leq \frac{1}{2} \tag{3.18}$$

$$\left.\begin{aligned}\psi_{30}(t) &= \sqrt{\frac{8}{\pi}} \\ \psi_{31}(t) &= \frac{4}{\sqrt{\pi}}(8t-5) \\ \psi_{32}(t) &= \frac{4}{\sqrt{\pi}}(2(8t-5)^2-1)\end{aligned}\right\} \frac{1}{2} \leq t \leq \frac{3}{4} \tag{3.19}$$

$$\left.\begin{aligned}\psi_{40}(t) &= \sqrt{\frac{8}{\pi}} \\ \psi_{41}(t) &= \frac{4}{\sqrt{\pi}}(8t-7) \\ \psi_{42}(t) &= \frac{4}{\sqrt{\pi}}(2(8t-7)^2-1)\end{aligned}\right\} \frac{3}{4} \leq t \leq 1 \tag{3.20}$$

## 3.4. Optimal Control Problem Reformulation

The linear quadratic optimal control problem can be stated as follows:
Find an optimal controller $u^*(t)$ that minimizes the following quadratic performance index

$$J = \int_0^{t_f} (x^T Q x + u^T R u) dt \tag{3.21}$$

*subject to*

$$\dot{x} = Ax + Bu \tag{3.22}$$

$$x(0) = x_0 \tag{3.23}$$

Because Chebyshev wavelets are defined on the time interval $\tau \in [0,1]$ and since our problem is defined on the interval $t \in [0, t_f]$ it is necessary before using Chebyshev

wavelets to transform the time interval of the optimal control problem into the interval $\tau \epsilon [0,1]$.
We can obtained that by using

$$\tau = \frac{t}{t_f} \quad (3.24)$$

So,

$$dt = t_f d\tau \quad (3.25)$$

Then the optimal control problem became as

$$J = t_f \int_0^1 (x^T Qx + u^T Ru) d\tau \quad (3.26)$$

$$\frac{dx}{d\tau} = t_f (Ax + Bu) \quad (3.27)$$

### 3.4.1. Control State Parameterization

The basic idea is to approximate the state and control variables by a finite series of Chebyshev wavelets as follows [20]

$$x_i(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} a^i{}_{nm} \phi_{nm}(t) \qquad i = 1,2, \ldots, s \quad (3.28)$$

$$u_i(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} b^i{}_{nm} \phi_{nm}(t) \qquad i = 1,2, \ldots, r \quad (3.29)$$

We can write these two equations in compact form as :

$$x(t) = \left(I_s \otimes \Phi^T(t)\right) a \quad (3.30)$$

$$u(t) = \left(I_r \otimes \Phi^T(t)\right) b \quad (3.31)$$

$where\ I_s, I_r$ are s x s and r x r identity matrices and $\Phi(t)$ is $N\ x\ 1$ ,

$N = 2^k(M), vector\ of\ Chebychev\ scaling\ function\ given\ by$ :

$$\Phi(t) = [\Phi_{1m-1}(t), \Phi_{2m-1}(t), \Phi_{3m-1}(t), \ldots, \phi_{2^k m-1}(t)]^T \quad (3.32)$$

$$\Phi_{im-1}(t) = [\phi_{i0}(t), \phi_{i1}(t), \ldots, \phi_{iM-1}(t)] \quad (3.33)$$

$and$

$$a = [\alpha^1\ \alpha^2 \ldots . \alpha^s]^T$$

$$\alpha^i = \left[a_{10}^i\ a_{11}^i \ldots . a_{1M-1}^i\ a_{20}^i \ldots . a_{2M-1}^i \ldots a_{2^k 0}^i \ldots .\ a_{2^k M-1}^i\right] \quad i = 1,2, \ldots, s \quad (3.34)$$

$$b = [\beta^1 \ \beta^2 \ \ldots. \beta^r]^T$$
$$\beta^i \ = \left[b_{10}^i \ \ b_{11}^i \ \ldots. b_{1M-1}^i \ \ b_{1M-1}^i \ldots. b_{2M-1}^i \ldots b_{2^k 0}^i \ \ldots. \ \ b_{2^k M-1}^i\right] \quad i = 1,2,\ldots,r \qquad (3.35)$$

*where $a$ , $b$ are vectors of unknown parameters of dimensions $sN \ x \ 1$ and $rN \ x \ 1$.*

To approximate the state equation via Chebyshev scaling functions equation ( 3.22 ) can be integrated as

$$x(t) - x_o = \int_0^t Ax(\tau)d\tau + \int_0^t Bu(\tau)d\tau \qquad (3.36)$$

### 3.4.2. Initial Condition

The initial condition vector $x_o$ can be expressed via Chebyshev scaling function as

$$\begin{aligned} x_o \ &= \ \frac{\sqrt{\pi/2}}{2^{k/2}}(I_s \otimes \Phi^T(t))[\alpha_0^1 \ \alpha_0^2 \ \ldots. \alpha_0^s] \\ &= \ \frac{\sqrt{\pi/2}}{2^{k/2}}\left(I_s \otimes \Phi^T(t)\right)g_o \end{aligned} \qquad (3.37)$$

*where* $g_o = [\alpha_0^1 \ \alpha_0^2 \ \ldots. \alpha_0^s]$ *and* $\alpha_0^i = [x_i(0) \ 0 \ \ 0 \ldots 0 \ \ \ x_i(0) \ 0 \ \ 0 \ldots 0 \ldots \ x_i(0) \ 0 \ \ 0 \ldots 0]$

*We multiply Eq.* ( 3.37 ) *by factor,*

$$\delta = \frac{\sqrt{\frac{\pi}{2}}}{2^{\frac{k}{2}}}$$

*because from Eq.* ( 3.4 ) *we can obtained*

$$\phi_{n0} = \frac{2^{k/2}}{\sqrt{\pi/2}}$$

*By substituting Eq.* ( 3.30), ( 3.31) *and* ( 3.37 ) *into* ( 3.36 ) *and using the operational matrix ,we get*

$$\left(I_s \otimes \phi^T(t)\right)a - \left(I_s \otimes \phi^T(t)\right)g_0\delta = A(I_s \otimes \phi^T(t)P^T)a + B(I_r \otimes \phi^T(t)P^T)b \qquad (3.38)$$

Using Kronecker product properties [ 35 ]we have

$$\begin{aligned} \left(I_s \otimes \phi^T(t)\right)a = \left(I_s \otimes \phi^T(t)\right)\left(A \otimes \mathrm{P}^T(t)\right)a + \left(I_s \otimes \phi^T(t)\right)\left(B \otimes P^T(t)\right)b \\ + \left(I_s \otimes \phi^T(t)\right)g_0\delta \end{aligned} \qquad (3.39)$$

By equating the coefficients of $\left(I_s \otimes \phi^T(t)\right)$ , we get

$$\left((A \otimes P^T) - I_{Ns}\right)a + \left(B \otimes P^T(t)\right)b + g_0\delta = 0 \qquad (3.40)$$

or

$$\left[(A\otimes P^T) - I_{Ns} \quad \left(B\otimes P^T(t)\right)\right]\begin{bmatrix} a \\ b \end{bmatrix} = -g_0\delta \tag{3.41}$$

$where\ I_{Ns}\ is\ Ns\ x\ Ns\ identity\ matrix.$

### 3.4.3. Performance Index Approximation

Then we substitute ( 3.30 ) and ( 3.31) into ( 3.21 ) to get [8]

$$J = \int_0^1 (a^T(\left(I_s\otimes\Phi(t)\right)Q\left(I_s\otimes\phi^T(t)\right)a + b^T\left(I_r\otimes\Phi(t)\right)R\left(I_r\otimes\phi^T(t)\right)b)dt \tag{3.42}$$

Then we simplified it as

$$J = \int_0^1 (a^T\,(Q\otimes\Phi(t)\phi^T)a + b^T(R\otimes\Phi(t)\phi^T)b)dt \tag{3.43}$$

Because of the orthogonality of Chebyshev scaling functions and from Lemma1 then we have :

$$\int_0^1 \Phi(t)\Phi^T(t)dt = RR$$

Then

$$J = a^T(Q\otimes RR)a + b^T(R\otimes RR)b \tag{3.44}$$

Finally we can write it as

$$J = [\ a^T \quad b^T\ ]\begin{bmatrix} Q\otimes RR & 0_{N_s\,x\,N_r} \\ 0_{N_r\,x\,N_s} & R\otimes RR \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \tag{3.45}$$

### 3.4.4. Continuity of the State Variables

To insure the continuity of the state variables between the different sections we must add constraints. There are $2^k - 1$ points at which the continuity of the state variables have to ensured.

Theses points are :

$$t_i = \frac{i}{2^k} \qquad\qquad i = 1,2,\ldots,2^k - 1 \tag{3.46}$$

So there are $(2^k - 1)s\ \ equality\ constraints\ given\ by$ :

$$\left(I_s\otimes\Phi'(t)\right)a = 0_{(2^k-1)s\,x\,1} \tag{3.47}$$

Where

$$\Phi' = \begin{bmatrix} \phi_{1m-1}(t_1) & -\phi_{2m-1}(t_1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & \phi_{2m-1}(t_2) & -\phi_{3m-1}(t_2) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \phi_{3m}(t_3) & -\phi_{4m}(t_3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \phi_{(2^k-1)m}\left(t_{2^k-1}\right) & -\phi_{(2^k-1)m}\left(t_{2^k-1}\right) \end{bmatrix}$$

( 3.48 )

$\Phi' is\ (2^k - 1)\ x\ (2^k\text{M})$ matrix

### 3.4.5. Quadratic Optimal Control Transformation

By combining the equality constraints ( 3.41) with those in ( 3.47)we have

$$\begin{bmatrix} (A \otimes P^T) - I_{Ns} & (B \otimes P^T) \\ (I_s \otimes \Phi') & 0_{(2^k-1)s\ x\ Nr} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -g_0\delta \\ 0_{(2^k-1)s\ x\ 1} \end{bmatrix} \quad (3.49)$$

From ( 3.45 ) and ( 3.49 ) the optimal control problem is transformed into the following quadratic programming problem

$$\min_z z^T H z \quad (3.50)$$

Subject to equality constraints

$$Fz = h \quad (3.51)$$

*where*

$$z^T = [\ a^T \quad b^T] \quad (3.52)$$

$$H = \begin{bmatrix} Q \otimes RR & 0_{N_s\ x\ N_r} \\ 0_{N_r\ x\ N_s} & R \otimes RR \end{bmatrix} \quad (3.53)$$

$$\text{F} = \begin{bmatrix} (A \otimes P^T) - I_{Ns} & (B \otimes P^T) \\ (I_s \otimes \Phi') & 0_{(2^k-1)s\ x\ Nr} \end{bmatrix} \quad 3.54)$$

$$\text{h} = \begin{bmatrix} -g_0\delta \\ 0_{(2^k-1)s\ x\ 1} \end{bmatrix} \quad (3.55)$$

## 3.5. Numerical Example 1

### Problem Treated by Feldbaum

Find the optimal control $u^*(t)$ which minimizes

$$J = \frac{1}{2}\int_0^1 (x^2 + u^2)dt$$

subject to

$$\dot{x} = -x + u \quad , \quad x(0) = 1$$

We solved this problem when $k = 1\ , and\ M = 3\ , so\ N = 6$

Then we approximate the state and control variables as

$$x(t) = \sum_{n=1}^{2}\sum_{m=0}^{2} a_{nm}\phi_{nm}(t) \qquad (3.56)$$

$$\mathrm{u}(t) = \sum_{n=1}^{2}\sum_{m=0}^{2} \mathrm{b}_{nm}\phi_{nm}(t) \qquad (3.57)$$

*For this problem*

*Chebyshev scaling functions for this problem are for k=1,M=3*

$$\left.\begin{array}{l}\psi_{10}(t) = \frac{2}{\sqrt{\pi}} \\ \psi_{11}(t) = \frac{2\sqrt{2}}{\sqrt{\pi}}(4t-1) \\ \psi_{12}(t) = \frac{2\sqrt{2}}{\sqrt{\pi}}(2(4t-1)^2-1)\end{array}\right\} \qquad \left.\begin{array}{l}\psi_{20}(t) = \frac{2}{\sqrt{\pi}} \\ \psi_{21}(t) = \frac{2\sqrt{2}}{\sqrt{\pi}}(4t-3) \\ \psi_{22}(t) = \frac{2\sqrt{2}}{\sqrt{\pi}}(2(4t-3)^2-1\end{array}\right\}$$

$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \psi_{12}(t), \psi_{20}(t), \psi_{21}(t), \psi_{22}(t)]^T$

$a = [a_{10}(t), a_{11}(t), a_{12}(t), a_{20}(t), a_{21}(t), a_{22}(t)]$

$b = [b_{10}(t), b_{11}(t), b_{12}(t), b_{20}(t), b_{21}(t), b_{22}(t)]$

$\delta = \frac{\sqrt{\pi/2}}{2^{k/2}} = \frac{\sqrt{\pi/2}}{2}$

$g_0 = [1\ 0\ 0\ 1\ 0\ 0]$

There are $2^k - 1 = 1$ point.

This point is :

$$t_1 = \frac{1}{2^k} = 0.5 \qquad i = 1$$

So there are $(2^k-1)s = 1$ $equality\ constraint\ given\ by$ :

$$(I_s \otimes \Phi'(t))a = 0_{(2^k-1)s\ x\ 1}$$

$\Phi' is$ $(2^k-1)\ x\ (2^k(\mathrm{M}))$ matrix then $[\Phi']_{1x6}$ matrix

$\Phi' = [\psi_{10}(0.5), \psi_{11}(0.5), \psi_{12}(0.5), -\psi_{20}(0.5), -\psi_{21}(0.5), -\psi_{22}(0.5)]$
$\Phi' =$ [1.1284 1.5958 1.5958 -1.1284 +1.5958 -1.5958]

By solving the corresponding quadratic programming problem we obtained the optimal value of performance index

$$(J = 0.193001037554299) \qquad for\ k = 1, M = 3$$

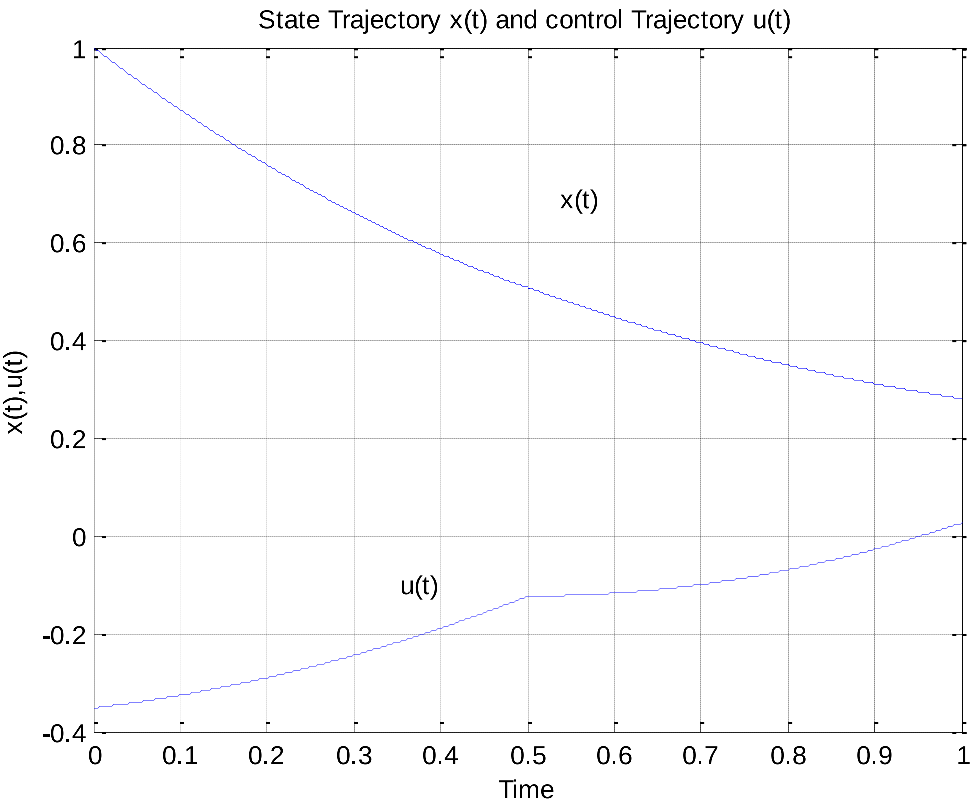


**Figure ( 3.1 )** Optimal state and control trajectories $x(t)$ $and$ $u(t)$ $k = 1, M = 3$

$$(J \;=\; 0.192915719226705) \qquad for\; k = 2\,, M = 3$$

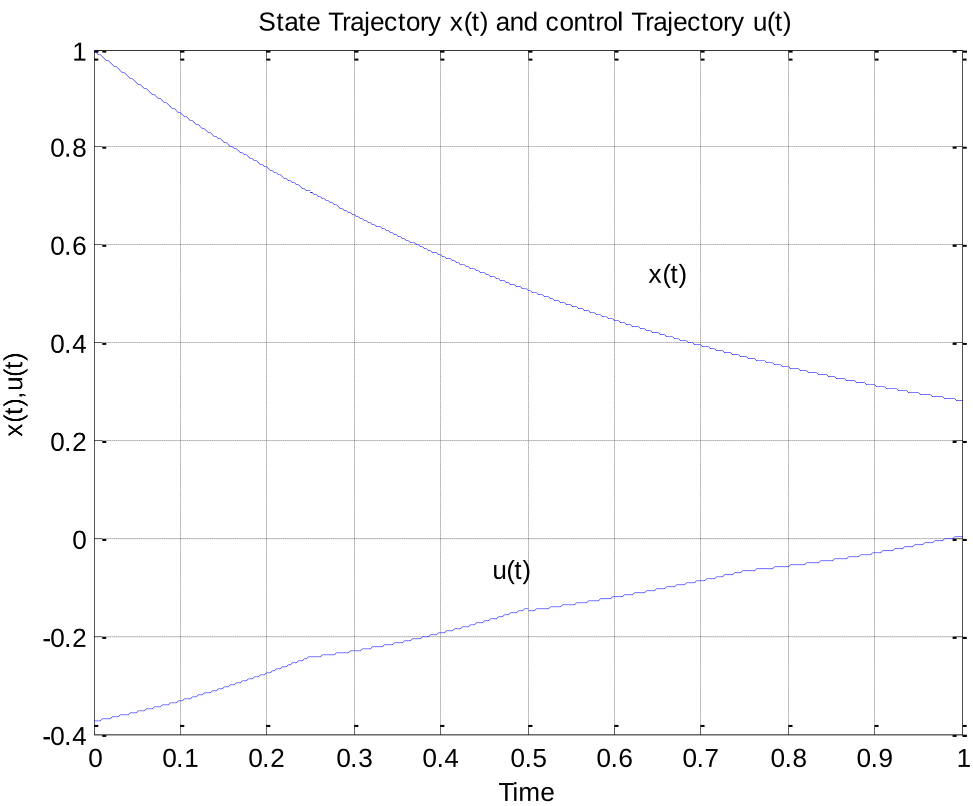


**Figure ( 3. 2 )** Optimal state and control trajectories $x(t)$ $and$ $u(t)$ $k = 2, M = 3$

$$(J = 0.192909783507572) \quad for\ k = 3, M = 3$$

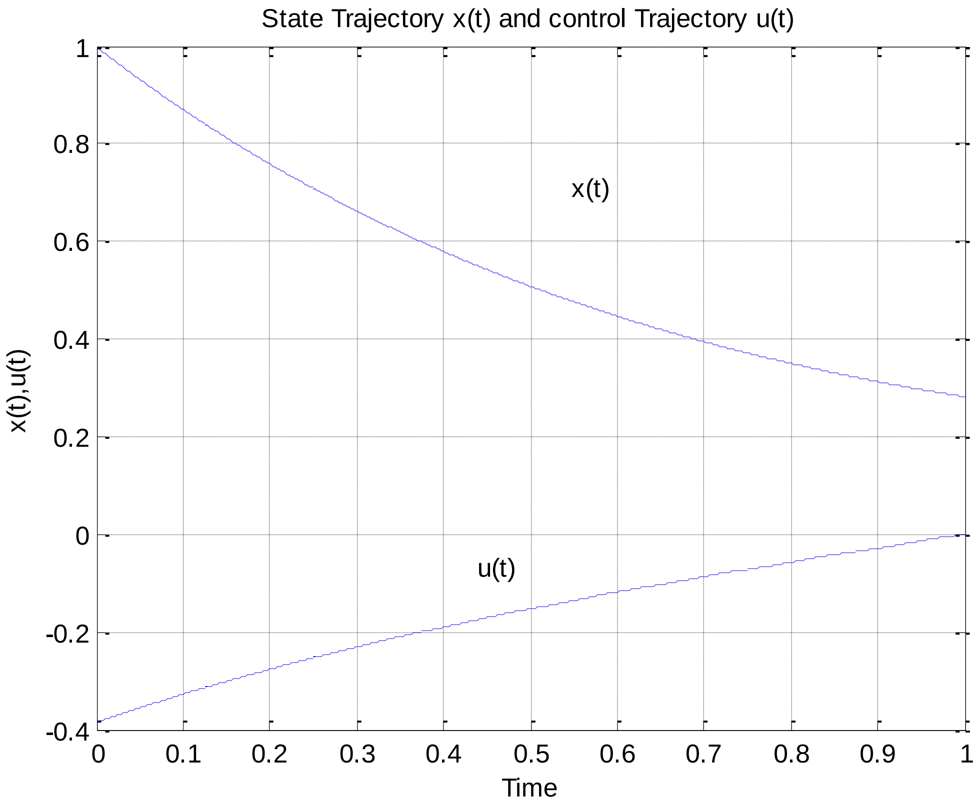


**Figure ( 3.3 )** Optimal state and control trajectories $x(t)\ and\ u(t)\ k = 3, M = 3$

**Table** (3.1)

| | $K = 1,$ $M = 3$ | $K = 2,$ $M = 3$ | $K = 3,$ $M = 3$ | $K = 3,$ $M = 4$ | $Exact\ value$ |
|---|---|---|---|---|---|
| $J$ | 0.1930010375 | 0.1929157192 | 0.1929097835 | 0.1929093208 | 0.1929092981 |

We conclude from Table ( 3.1 ) that when we increase k or M we can obtain the results of performance index ( J ) more closed to the exact value. .

Also from Figures ( 3.1 – 3.3 ) we conclude that, we can plot the OCP trajectories more good when we increase in K and M.

### 3.6. Numerical Example 2

Find an optimal controller $u(t)$ that minimizes the following performance index

$$J = \frac{1}{2}\int_0^1 (x_1^2 + x_2^2 + 0.005u^2)dt$$

subject to

$$\dot{x}_1 = x_2 \qquad\qquad x_1(0) = 0$$

$$\dot{x}_2 = -x_2 + u \qquad\qquad x_2(0) = -1$$

We apply the proposed method at this example , we solved this problem when

$k = 3\,, and\ M = 5 \qquad J = 0.0694046775616713$

$k = 3\,, and\ M = 6 \qquad J = 0.0693859107633072$

By solving the corresponding quadratic programming problem we obtained the optimal value of performance index ( $J = 0.0693859107633072$ ), while the exact value is ( $J = 0.06936094$ ) .

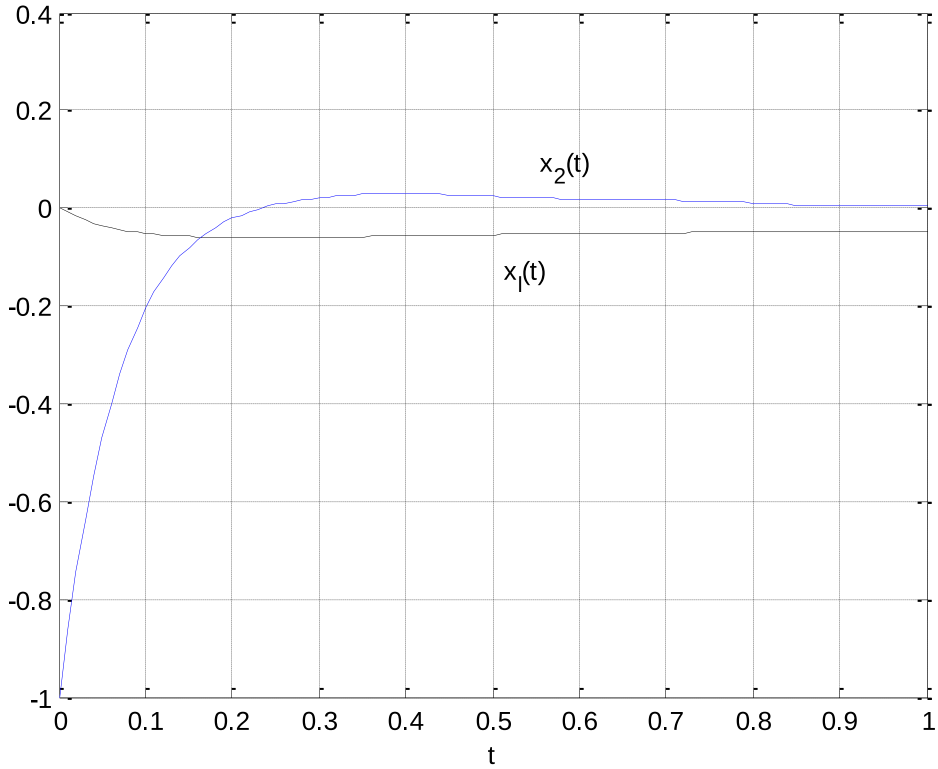


**Figure ( 3.2 )** *Optimal state trajectories* $x_1(t) and\ x_2(t)$

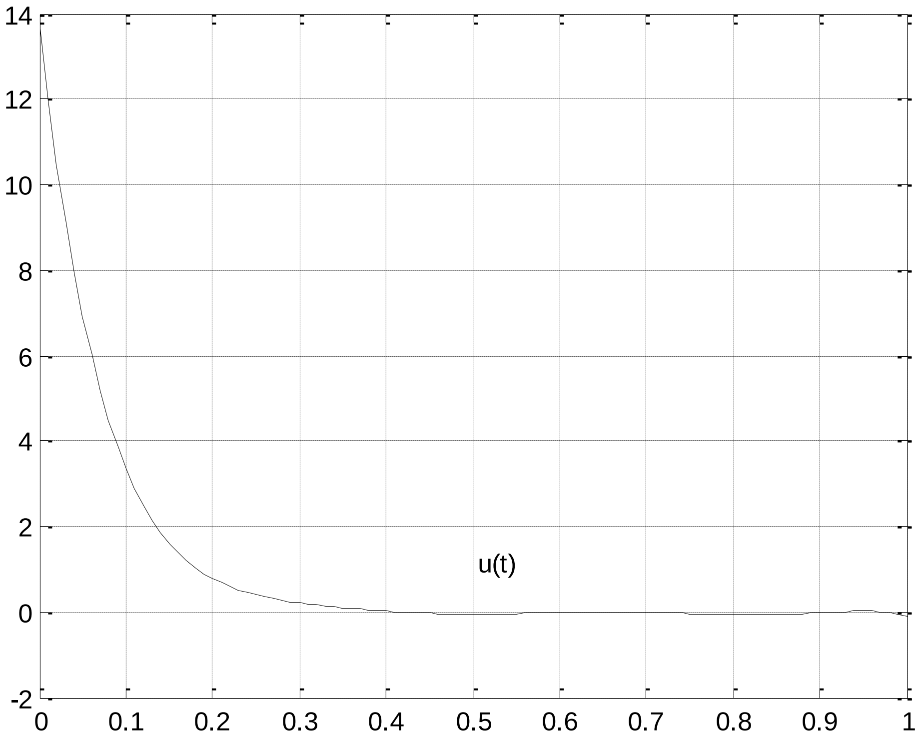


**Figure ( 3.3 )** *optimal control trajectory* $u(t)$

**Table ( 3.1 )** Comparison between different researches for ( $J$ ) value

| Research Name | $J$ | Deviation error |
|---|---|---|
| Exact value | 0.06936094 | 0 |
| Hsieh [36] | 0.0702 | $8.4\ x\ 10^{-4}$ |
| Neuman and Sen [31] | 0.06989 | $5.3\ x\ 10^{-4}$ |
| Vlassenbroeck [41] | 0.069368 | $7.1\ x\ 10^{-6}$ |
| Jaddu [2] | 0.0693689 | $7.96\ x\ 10^{-6}$ |
| Majdalawi [22] | 0.0693668896 | $7.9562x\ 10^{-6}$ |
| This research | 0.0693859107 | $2.49x\ 10^{-5}$ |

In this chapter, we proposed a numerical method for solving linear time in-variant quadratic optimal control problems. In this method we used Chebyshev wavelet to approximate controls and states of the system using a finite length of Chebyshev wavelet.

Then we solved two examples, the first example contains one state and the second example contains two states, compared with other researches, our research gives better or comparable results with other researches.

As we saw in this chapter we converted the difficult linear quadratic optimal control problem into a quadratic programming problem which was easy to solve, and solved it by MATLAB program.

# CHAPTER 4 OPTIMAL CONTROL PROBLEM OF LINEAR TIME-VARYING SYSTEMS

In this chapter we present Chebyshev scaling function multiplication formula and multiplication operational matrix. A numerical method is presented to solve the time-varying linear optimal control problem. The method is based on converting the optimal control problem into a quadratic programming problem .

Because we aimed to solve the nonlinear optimal control problem, so we want to use the iteration technique which developed by Banks [10-13] which replaces the original nonlinear dynamic state equations by an equivalent sequence of linear time-varying state equations. By this, the original nonlinear quadratic optimal control problem is converted into a sequence of quadratic linear time-varying optimal control problems which are much easier to solve, we will see that in next chapter.

## 4.1. Statement of the Optimal Control of Linear Time-Varying Systems

Find the optimal control that minimizes the quadratic performance index

$$\boldsymbol{J} = \int_{\mathbf{0}}^{t_f} (\boldsymbol{x^T Q x} + \boldsymbol{u^T R u})\boldsymbol{dt} \qquad (\mathbf{4.1})$$

Subject to the time-varying system given by

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t}), \qquad \boldsymbol{x}(\mathbf{0}) = \boldsymbol{x_0} \qquad (\mathbf{4.2})$$

where $x \in R^s$ *is the state variables vector*, $u \in R^r$ *is the control vector* , $x_o \in R^s$ *is the vector of initial conditions*, $A(t)$ *and* $B(t)$ are time-varying matrices, Q is a positive semidefinite matrix , and R is a positive definite matrix .

## 4.2. Optimal Control Problem

### 4.2.1. Control State Parameterization

Approximating the state variables and the control variables by Chebyshev scaling functions, we get [20]

$$x_i(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} a^i{}_{nm} \phi_{nm}(t) \qquad i = 1,2, \dots, s \qquad (4.3)$$

$$u_i(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} b^i{}_{nm} \phi_{nm}(t) \qquad i = 1,2, \dots, r \qquad (4.4)$$

We can write these two equations in compact form as :

$$x(t) = (\Phi^T(t) \otimes I_s) a \qquad (4.5)$$

$$u(t) = (\Phi^T(t) \otimes I_r) b \qquad (4.6)$$

Where $I_s, I_r$ *are* $s \ x \ s$ *and* $r \ x \ r$ *identity matrices respectively*, $\Phi^T(t)$ *is* $N \ x \ 1$, $(N = 2^k(M))$, *vector of Chebyshev scaling function given by* :

$$\Phi(t) = [\Phi_{1m-1}(t), \Phi_{2m-1}(t), \Phi_{3m-1}(t), \dots, \phi_{2^k m-1}(t)]^T \qquad (4.7)$$

$$\Phi_{im-1}(t) = [\phi_{i0}(t), \phi_{i1}(t), \dots, \phi_{iM-1}(t)] \qquad (4.8)$$

*and*

$$a = [\alpha^1 \ \alpha^2 \dots . \alpha^s]^T \qquad (4.9)$$

$$\alpha^i = \left[a^i_{10} \ a^i_{11} \dots . a^i_{1M-1} \ a^i_{20} \dots . a^i_{2M-1} \dots a^i_{2^k 0} \dots . \ a^i_{2^k M-1}\right] \quad i = 1,2, \dots, s \qquad (4.10)$$

$$b = [\beta^1 \ \beta^2 \dots . \beta^r]^T \qquad (4.11)$$

$$\beta^i = \left[b^i_{10} \ b^i_{11} \dots . b^i_{1M-1} \ b^i_{1M-1} \dots . b^i_{2M-1} \dots b^i_{2^k 0} \dots . \ b^i_{2^k M-1}\right] \quad i = 1,2, \dots, r \qquad (4.12)$$

$a$ *and* $b$ are vectors of unknown parameters have dimensions $sN \ x \ 1$ *and* $rN \ x \ 1$ respectively .

### 4.2.2. The Product Operational Matrix of Chebyshev Wavelets

The following property of the product of two Chebyshev wavelets vectors [20] will also be used. Let

$$\Psi(t)\Psi^T(t)F = \breve{F}\Psi(t), \qquad (4.13)$$

*Where*

$\tilde{F}$ *is* $(2^k M) x \ (2^k M)$ *matrix. To illustrate the calculation procedure we choose* $M = 3$ *and* $k = 2$.

*Thus we have*:

$$F = [f_{10}, f_{11}, f_{12}, \dots, f_{40}, f_{41}, f_{42}]^T$$

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \psi_{12}(t), \dots, \psi_{4,0}(t), \psi_{41}(t), \psi_{42}(t)]^T$$

$Then$

$$\tilde{F}=\begin{bmatrix}\tilde{F}_1 & 0 & 0 & 0\\ 0 & \tilde{F}_2 & 0 & 0\\ 0 & 0 & \tilde{F}_3 & 0\\ 0 & 0 & 0 & \tilde{F}_4\end{bmatrix}$$

$In\ general\ case\ \tilde{F}\ \ is\ a\ (\ 2^kM)\ \ X\ \ (\ 2^kM)$

$$\tilde{F}=\begin{bmatrix}\tilde{F}_1 & 0 & \cdots & 0\\ 0 & \tilde{F}_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \tilde{F}_{2^k}\end{bmatrix} \qquad (4.14)$$

$Where$

$$\tilde{F}_i = \frac{2^{\frac{k}{2}}}{\sqrt{\pi}}\begin{bmatrix} f_{i0} & f_{i1} & f_{i2} & f_{i3} & \cdots & f_{i,M-2} & f_{i,M-1}\\ f_{i1} & f_{i0}+\frac{1}{\sqrt{2}}f_{i2} & \frac{1}{\sqrt{2}}(f_{i1}+f_{i3}) & \frac{1}{\sqrt{2}}(f_{i2}+f_{i4}) & \cdots & \frac{1}{\sqrt{2}}(f_{i,M-3}+f_{i,M-1}) & \frac{1}{\sqrt{2}}f_{i,M-2}\\ f_{i2} & \frac{1}{\sqrt{2}}(f_{i1}+f_{i3}) & f_{i0}+\frac{1}{\sqrt{2}}f_{i4} & \frac{1}{\sqrt{2}}(f_{i1}+f_{i5}) & \cdots & \frac{1}{\sqrt{2}}f_{i,M-4} & \frac{1}{\sqrt{2}}f_{i,M-3}\\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots\\ \cdots & \cdots & \cdots & f_{i0}+\frac{1}{\sqrt{2}}f_{i,\mu} & f_{i1}+\frac{1}{\sqrt{2}}f_{i,\mu+1} & \cdots & \frac{1}{\sqrt{2}}f_{i,\nu}\\ \cdots & \cdots & \cdots & f_{i1}+\frac{1}{\sqrt{2}}f_{i,\mu+1} & f_{i0} & \vdots & \vdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & f_{i0} & \frac{1}{\sqrt{2}}f_{i1}\\ f_{i,M-1} & \frac{1}{\sqrt{2}}f_{i,M-2} & \cdots & \cdots & \cdots & \frac{1}{\sqrt{2}}f_{i1} & f_{i0}\end{bmatrix}$$

$$\mu=\begin{cases}M-2 & M\ even\\ M-1 & M\ odd\end{cases}$$

$$\nu=\begin{cases}M/2 & M\ even\\ \dfrac{M-1}{2} & M\ odd\end{cases} \qquad (4.15)$$

### 4.2.3. Performance Index Approximation

To approximate the performance index, we substitute $Eq.$ ( 4.5 ) and ( 4.6) into ( 4.1 ) to get [39]

$$J=\int_0^1 (a^T\,(\Phi(t)\otimes I_s)Q(\Phi^T(t)\otimes I_s)a+b^T(\Phi(t)\otimes I_r)R(\Phi^T(t)\otimes I_r)b)dt \qquad (4.16)$$

It can be simplified as

$$J = \int_0^1 (a^T\,(\Phi(t)\Phi^T(t) \otimes Q)a + b^T(\Phi(t)\Phi^T(t) \otimes R)b)dt \tag{4.17}$$

Because of orthogonality of Chebyshev scaling function and using Lemma1 in chapter three

$$\int_0^1 \Phi(t)\Phi^T(t)dt = RR \tag{4.18}$$

Then

$$J = a^T(RR \otimes Q)a + b^T(RR \otimes R)b \tag{4.19}$$

It can be wrote as

$$J = [\ a^T \quad b^T\ ]\begin{bmatrix} RR \otimes Q & 0_{N_s\,x\,N_r} \\ 0_{N_r\,x\,N_s} & RR \otimes R \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \tag{4.20}$$

To approximate the state equations we write equation ( 4.5 ) as

$$x = \sum_{i=1}^{2^k}\sum_{j=0}^{M-1} \phi_{ij}(t)\,\alpha_{ij} \tag{4.21}$$

$Or$

$$x = \Phi^T(t)\big[\alpha_{10}\alpha_{11} \dots \alpha_{1M-1}\alpha_{20}\ \dots\ \alpha_{2M-1}\alpha_{2^k0}\ \dots\ \alpha_{2^kM-1}\ \big]^T$$

$$= \Phi^T(t)\alpha \tag{4.22}$$

$Where$ $\alpha_{ij} = [\mathrm{a}_{ij}{}^1\ \mathrm{a}_{ij}{}^2 \dots \mathrm{a}_{ij}{}^s]$

The control variables ( 4.6 ) can be rewritten as

$$u = \sum_{i=1}^{2^k}\sum_{j=0}^{M-1} \phi_{ij}(t)\,\beta_{ij} \tag{4.23}$$

$Or$

$$u = \Phi^T(t)\big[\beta_{10}\beta_{11} \dots \beta_{1M-1}\beta_{20}\ \dots\ \beta_{2M-1}\beta_{2^k0}\ \dots\ \beta_{2^kM-1}\ \big]^T$$

$$= \Phi^T(t)\beta \tag{4.24}$$

$Where$ $\beta_{ij} = [b_{ij}{}^1\ b_{ij}{}^2 \dots b_{ij}{}^r]$

### 4.2.4. Time Varying Elements Approximation

Then we need to express $A(t)$ $and$ $B(t)$ in terms of Chebyshev scaling functions. The approximation of $A(t)$ can be given by [38] :

$$A(t) = \sum_{i=1}^{2^k} \sum_{j=0}^{M-1} A_{ij} \phi_{ij}(t) \qquad (4.25)$$

$$A(t) = \left[A_{10} \quad A_{11} \dots A_{1M-1} \quad A_{20} \dots A_{2M-1} \dots A_{2^k 0} \dots A_{2^k M-1}\right] \Phi(t) \qquad (4.26)$$

$Where$
$A_{ij}$ $is\ an\ s\ x\ s\ constant\ matrix\ of\ the\ coefficents\ of\ Chebyshev\ scaling\ function$ $\phi_{ij}(t)$. Theses constant matrices can be obtained as

$$A_{ij} = \int_{\frac{i-1}{2^k}}^{\frac{i}{2^k}} A(t) \phi_{ij}(t) dt \qquad (4.27)$$

Similarly , $B(t)$ can be expanded via Chebyshev scaling functions as follows

$$B(t) = \left[B_{10} \quad B_{11} \dots B_{1M-1} \quad B_{20} \dots B_{2M-1} \dots B_{2^k 0} \dots B_{2^k M-1}\right] \Phi(t) \qquad (4.28)$$

$Where\ B_{ij}\ is\ an\ \ s\ x\ r\ constant\ matrix$

### 4.2.5. Initial Condition

The initial condition vector $x_o$ can be expressed via Chebyshev scaling function as

$$x_o = \frac{\sqrt{\pi/2}}{2^{k/2}} (\Phi^T(t)) [\alpha_0^1 \ \alpha_0^2 \ \dots . \alpha_0^s]$$

$$= \frac{\sqrt{\pi/2}}{2^{k/2}} \left(\Phi^T(t)\right) g_o \qquad (4.29)$$

$where$ $$g_o = \left[\alpha_{10}^0 \quad 0 \dots 0 \ \alpha_{20}^0 \ 0 \dots 0 \dots \alpha_{2^k 0}^0 \ 0 \dots 0\right]^T$$
$$and\ \ \alpha_{i0}^0 = [x_i(0) \quad x_2(0) \dots \ x_s(0)]$$

$We\ multiply\ Eq. (\ 4.29\ ) by\ factor,$

$$\delta = \frac{\sqrt{\frac{\pi}{2}}}{2^{\frac{k}{2}}}$$

$because\ from\ Eq. (\ 3.4\ ) we\ can\ obtained$

$$\Phi_{n0} = \frac{2^{k/2}}{\sqrt{\pi/2}}$$

To express the state equations in terms of the unknown parameters of the state variables and the control variables , $Eq.$ ( 4.2 ) can be integrated as

$$x(t) - x_0 = \int_0^t A(t)x(\tau)d\tau + \int_0^t B(t)u(\tau)d\tau \qquad (4.30)$$

*By substituting* ( 4.22 ) , ( 4.24) , ( 4.26 ) , ( 4.28 ) *and* ( 4.29 ) *into* ( 4.30 ) , *we get*

$$\Phi^T(t)\alpha - \Phi^T(t)\delta g_o = \int_0^t [A_{10} \ \dots \ A_{2^k M-1}] \, \Phi(t)\Phi^T(t)\alpha dt + \int_0^t [B_{10} \ \dots \ B_{2^k M-1}] \, \Phi(t)\Phi^T(t)\beta dt \qquad (4.31)$$

*But from* ( 4.13 ) *we have*

$$[A_{10} \ \dots \ A_{2^k M-1}] \, \Phi(t)\Phi^T = \Phi^T \tilde{A} \qquad (4.32)$$

$$[B_{10} \ \dots \ B_{2^k M-1}] \, \Phi(t)\Phi^T = \Phi^T \tilde{B} \qquad (4.33)$$

*where* $\tilde{A}$ *and* $\tilde{B}$ *are* $sN$ x $sN$ *and* $sN$ x $rN$ *constant matrices respectively. Substituting* ( 4.32 ) *and* ( 4.33 ) *into equation* ( 4.31 ) *gives*

$$\Phi^T(t)\alpha - \Phi^T(t)\delta g_o = \int_0^t \Phi^T \tilde{A}(t)\alpha dt + \int_0^t \Phi^T(t) \ \tilde{B}\beta dt \qquad (4.34)$$

Using the integration operational matrix ***P*** of Chebyshev scaling function , we get

$$\Phi^T(t)\alpha - \Phi^T(t)\delta g_o = \Phi^T(t)P^T\tilde{A}\alpha + \Phi^T(t)P^T\tilde{B}\beta \qquad (4.35)$$

$$(\Phi^T(t)\otimes I_s)a - (\Phi^T(t)\otimes I_s)\delta g_o = (\Phi^T(t)P^T\otimes I_s)\tilde{A}a + (\Phi^T(t)P^T\otimes I_s)\tilde{B}b \qquad (4.36)$$

$$I_{Ns}a - \delta g_o = (P^T\otimes I_s)\tilde{A}a + (P^T\otimes I_s)\tilde{B}b \qquad (4.37)$$

### 4.2.6. Quadratic Programming Problem Transformation

Finally by combining the equality constraints ( 4.37 ) with ( 3.47 ) we get

$$\begin{bmatrix} (P^T \otimes \mathrm{I_s})\tilde{A} - I_{Ns} & (P^T \otimes I_s)\tilde{\mathrm{B}} \\ (\Phi' \otimes \mathrm{I_s}) & 0_{(2^k-1)s \ x \ Nr} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -g_0\delta \\ 0_{(2^k-1)s \ x \ 1} \end{bmatrix} \qquad (4.38)$$

We saw that the optimal control problem is converted into a quadratic programming problem of minimizing the quadratic function ( 4.20 ) subject to the linear constraints ( 4.38 ) and solved it using MATLAB program.

## 4.3. Numerical Example

Find the optimal control $u(t)$ which minimizes

$$J = \frac{1}{2}\int_0^1 (x^2 + u^2)dt$$

subject to

$$\dot{x} = tx + u \qquad\qquad x(0) = 1$$

$We\ solved\ this\ problem\ for$

$k = 2\ and\ M = 3,4,5\ , the\ optimal\ value\ we\ get\ as\ in\ Table\ (4.1)\ as\ shown$

**Table** ( 4.1 )

| | $K = 2,$ $M = 3$ | $K = 2,$ $M = 4$ | $K = 2,$ $M = 5$ |
|---|---|---|---|
| $J$ | 0.484823598604444 | 0.484268435061873 | 0.484267810538982 |

The optimal state and control variables are shown in Figures ( 4.1-4.3 ), we noticed from Figures ( 4.1 – 4.3 ) and from Table ( 4.1 ) that when we increase M we obtained at a good trajectories plots and at good results of performance index ( J ).

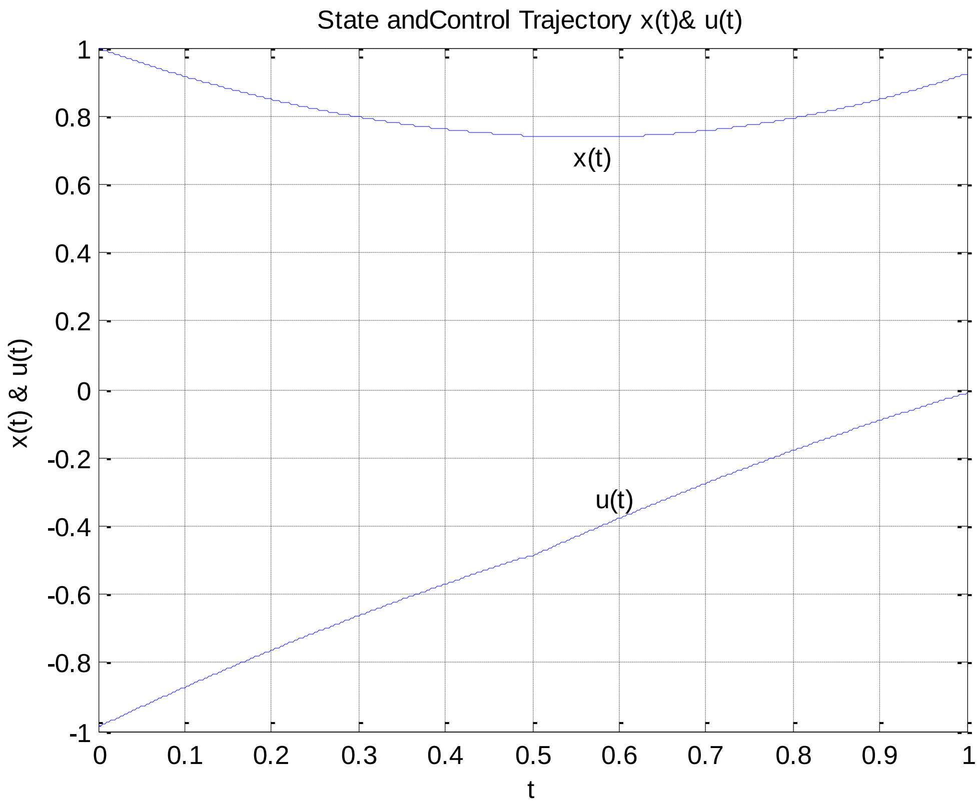


**Figure ( 4.1 )** Optimal state and control K=2 M=3

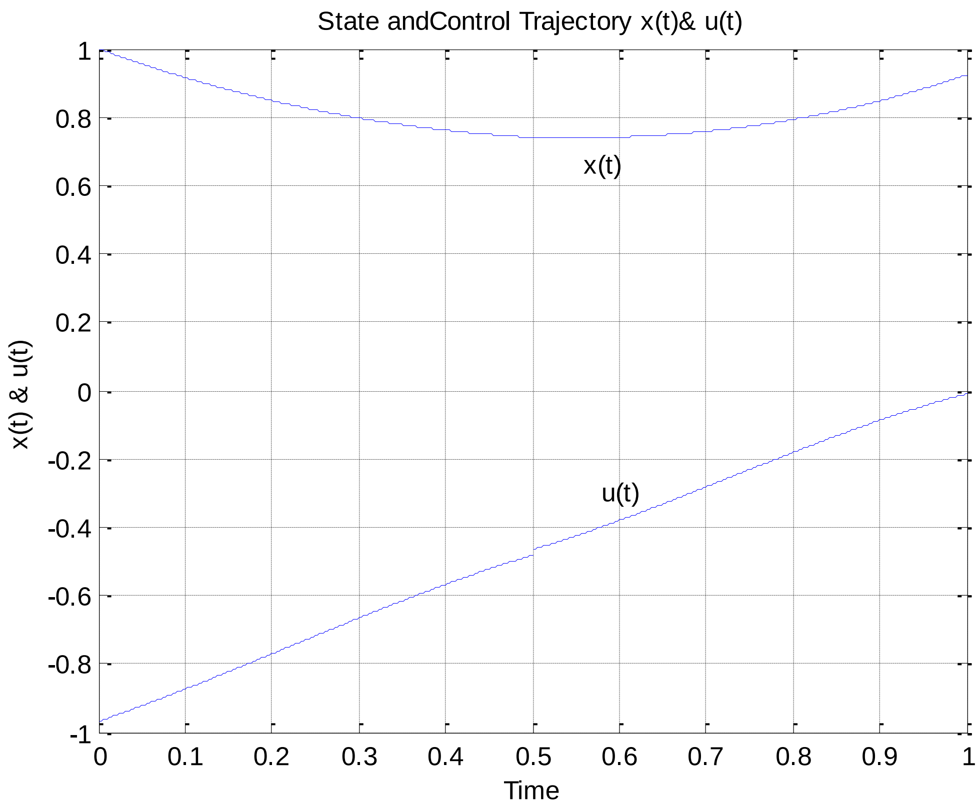


**Figure ( 4.2 )** Optimal state and control K=2 M=4

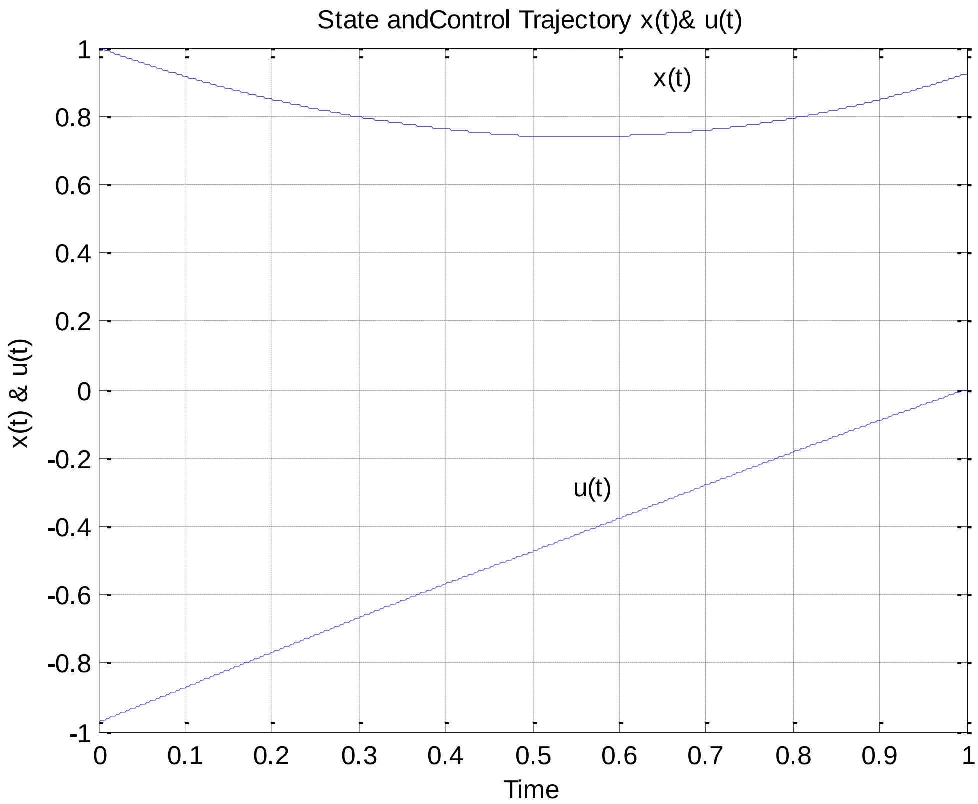


**Figure ( 4.3 )** Optimal state and control K=2 M=5

**Table** ( 4.2 )

| Research name | $Jaddu$[39] | $Elnajar$[40] | $This\ research$ |
|---|---|---|---|
| $J$ | 0.4842676003768 | 0.48427022 | 0.484267810538982 |

Table ( 4.2 ) shows the comparison between our research and other researches to solve the previous problem , from the table we notice that our method is good compared with other methods.

In this chapter we proposed a method to solve the optimal control problem time-varying systems using Chebyshev wavelet scaling function, we applied this method at a numerical example to see the effectiveness of the method and compared with other methods.

We need to solve the optimal control problem time-varying systems because we must need it to solve the nonlinear optimal control problem in the next chapter.

# CHAPTER 5 NONLINEAR QUADRATIC OPTIMAL CONTROL PROBLEM

## 5.1. Introduction

After we solved the linear quadratic optimal control problem in the previous chapters via Chebyshev wavelets, we look to solve the nonlinear optimal control problems also. I will use here the iteration technique developed by Banks [10-13] which replaces the original nonlinear dynamic state equations into an equivalent sequence of linear time-varying state equations, so the original nonlinear quadratic optimal control problem is converted into a sequence of quadratic linear time-varying optimal control problems which are much easier to solve.

Iteration technique is based on the replacement of the original nonlinear system by a sequence of linear time-varying systems, whose solutions will converge to the solution of the nonlinear problem. The only condition required for its application is Lipschitz condition which must be satisfied by a matrix associated with the nonlinear system. This approach will allow many of the classical results in linear systems theory to be applied to nonlinear systems [13].

Therefore, we extend the method described in chapter four to solve the nonlinear optimal control problem using Chebyshev wavelet.

Nonlinear optimal control problems was solved using parameterization methods have been published. Sirisena [40] used the piecewise polynomials to parameterize the control variables. Vlassenbroeck and Van Doreen [41] used the control-state parameterization using Chebyshev polynomials to convert the nonlinear optimal control problem into a nonlinear mathematical programming problem. In its turn, the nonlinear mathematical programming problem can then be solved using different methods. One of the popular methods that are used to solve the nonlinear mathematical programming problem is the sequential quadratic programming method [42] which replaces the nonlinear mathematical programming problem by a sequence of quadratic programming problems.

Jaddu [2],[43],[44] proposed a numerical method that is based on using the second method of quasilinearization and on parameterizing the system variables via Chebyshev polynomials to solve the nonlinear quadratic optimal control problems. By this, the original optimal control problem is converted directly into a quadratic programming problem.

Also Majdalawi [22] proposed a method that is based on state parameterization via Legendre polynomials which solved the nonlinear quadratic optimal control problems using iteration technique. So, the original optimal control problem is converted directly into a quadratic programming problem which is easy to solve.

As we say earlier, our method is based on replacing the difficult nonlinear dynamic system by a sequence of linear time-varying dynamic system using iteration technique [10-13]. These sequences of linear time-varying systems are to be solved using the method proposed in the previous chapter; which parameterize the control-state variables using Chebyshev wavelets.

## 5.2. Statement of the Nonlinear Quadratic Optimal Control Problem

The problem we are treating is to find the optimal control $u^*(t)$ that minimizes the performance index

$$\boldsymbol{J} = \int_{\boldsymbol{0}}^{\boldsymbol{t_f}} (\boldsymbol{x^T Q x} + \boldsymbol{u^T R u})\boldsymbol{dt} \qquad (\mathbf{5.1})$$

Subject to

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t}), \boldsymbol{t}) \qquad \boldsymbol{x}(\boldsymbol{0}) = \boldsymbol{x_0} \qquad (\mathbf{5.2})$$

Where $x \in R^n$, $u \in R^m$, $Q\ is\ n\ x\ n$ positive semidefinite matrix, $R\ is\ m\ x\ m$ positive definite matrix $x \in R^n\ and\ f$ is assumed continuous differentiable function with respect to all its arguments.

We proposed here a method that is based on using the iteration technique; in which the nonlinear dynamic system ( 5.1 ) is to be replaced by a sequence of linear time-varying dynamic system. So, the original nonlinear quadratic optimal control problem described in ( 5.1 ) – ( 5.2 ) is replaced by a sequence of linear quadratic optimal control problems that are easier to solve. The resulted linear quadratic time-varying optimal control problems are then to be solved using the method described in previous two chapters, then we combine between iteration technique and control-state parameterization via Chebyshev wavelets, the solution of the difficult nonlinear optimal control problem is reduced to a simple matrix-vector multiplication solved using MATLAB program.

## 5.3. What is Iteration Technique ?

Before starting this method we must know what is iteration technique. This technique is based on the replacement of the original nonlinear system by a sequence of linear time-varying systems, whose solutions will converge to the solution of the nonlinear problem.

Iteration technique was developed by Banks [10-13], in this technique, the nonlinear system described in ( 5.2 ) can be replaced by an equivalent sequence of linear time-varying state equations.

We can formulate this technique mathematically as follows
The nonlinear system in ( 5.2 ) can be rewritten in pseudo-linear form [45]:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} \quad , \quad \boldsymbol{x}(\boldsymbol{0}) = \boldsymbol{x_0} \tag{5.3}$$

The following sequence of linear time-varying state equations can replace the original nonlinear system described in ( 5.3 ):

$for\ k = 0$

$$\dot{\boldsymbol{x}}^{[\boldsymbol{0}]}(\boldsymbol{t}) = \boldsymbol{A}(\boldsymbol{x_0})\boldsymbol{x}^{[\boldsymbol{0}]} + \boldsymbol{B}(\boldsymbol{x_0})\boldsymbol{u}^{[0]} \quad , \boldsymbol{x}^{[\boldsymbol{0}]}(\boldsymbol{0}) = \boldsymbol{x_0} \tag{5.4}$$

$and\ for\ k \geq 1$

$$\dot{\boldsymbol{x}}^{[\boldsymbol{k}]}(\boldsymbol{t}) = \boldsymbol{A}\left(\boldsymbol{x}^{[\boldsymbol{k}-\boldsymbol{1}]}(\boldsymbol{t})\right)\boldsymbol{x}^{[\boldsymbol{k}]} + \boldsymbol{B}\left(\boldsymbol{x}^{[\boldsymbol{k}-\boldsymbol{1}]}(\boldsymbol{t})\right)\boldsymbol{u}^{[\boldsymbol{k}]} \quad , \boldsymbol{x}^{[\boldsymbol{k}]}(\boldsymbol{0}) = \boldsymbol{x_0} \tag{5.5}$$

First we must applied the Lipschitz condition at the above sequence. It can be shown that the above sequence converges to the solution of the original nonlinear system if the Lipschitz condition $\|A(x) - A(y)\| \leq \alpha\|x - y\|$ is satisfied. The proof can be found in [10].

If we applied the iteration technique to the optimal control problem described in (5.1) – (5.2), the following sequence of linear time-varying quadratic optimal control problems can replace the original problem in (5.1) $and$ (5.2):

$Minimize$

$$J^{[0]} = \int_0^{t_f} (x^{[0]^T} Q x^{[0]} + u^{[0]^T} R u^{[0]}) dt \tag{5.6}$$

$Subject\ to$

$$\dot{\boldsymbol{x}}^{[\boldsymbol{0}]}(\boldsymbol{t}) = \boldsymbol{A}(\boldsymbol{x_0})\boldsymbol{x}^{[\boldsymbol{0}]} + \boldsymbol{B}(\boldsymbol{x_0})\boldsymbol{u}^{[0]} \qquad \boldsymbol{x}^{[\boldsymbol{0}]}(\boldsymbol{0}) = \boldsymbol{x_0} \tag{5.7}$$

$And\ for\ k\ \geq\ 1$

$Minimize$

$$J^{[k]} = \int_0^{t_f} (x^{[k]^T} Q x^{[k]} + u^{[k]^T} R u^{[k]}) dt \tag{5.8}$$

$Subject\ to$

$$\dot{\boldsymbol{x}}^{[\boldsymbol{k}]}(\boldsymbol{t}) = \boldsymbol{A}\left(\boldsymbol{x}^{\boldsymbol{k}-\boldsymbol{1}}(\boldsymbol{t})\right)\boldsymbol{x}^{[k]} + \boldsymbol{B}\left(\boldsymbol{x}^{\boldsymbol{k}-\boldsymbol{1}}(\boldsymbol{t})\right)\boldsymbol{u}^{[\boldsymbol{k}]} \qquad \boldsymbol{x}^{[\boldsymbol{0}]}(\boldsymbol{0}) = \boldsymbol{x_0} \tag{5.9}$$

## 5.4. Problem Reformulation

The proposed method here converts the optimal control problem under consideration directly into a quadratic programming problem. To convert the optimal control problem (5.6) – (5.9) into a quadratic programming problem, the state and control variables are approximated by a finite length Chebyshev wavelets with unknown parameters. These approximations are used to approximate the initial state conditions of the system, which will be treated as linear constraints.

If we look at the optimal control problem OCP ( 5.6 ) – ( 5.9 ) we can easily show that the $0^{th}$ $iteration$ ( $k = 0$) problem ( 5.6 ) – ( 5.7 ) is a time-invariant optimal control problem, so we can consider this problem as the starting nominal trajectory to the sequence of optimal control problem ( 5.6 ) – ( 5.9 ). The solution to this particular problem was described in details in chapter three.

The remaining linear time-varying optimal control ( 5.8 ) – ( 5.9 ) can be solved using the method which was explained in chapter four using Chebyshev wavelets.

Because of Chebyshev wavelets are defined on the time interval $\tau \epsilon [0,1]$ and since our problem is defined on the interval $t \epsilon [0, t_f]$ it is necessary before using Chebyshev wavelets to transform the time interval of the optimal control problem into the interval $\tau \epsilon [0,1]$.
We can obtained that by using this formula,

$$\tau = \frac{t}{t_f} \qquad (5.10)$$

So,

$$dt = t_f d\tau \qquad (5.11)$$

Reformulate the OCP by transforming its time interval into $\tau \epsilon [0,1]$.
$for\ k \geq 1$

$Minimize$

$$J^{[k]} = t_f \int_0^1 (x^{[k]^T} Q x^{[k]} + u^{[k]^T} R u^{[k]}) d\tau \qquad (5.12)$$

$Subject\ to$

$$\boldsymbol{\frac{dx^{[k]}}{d\tau} = t_f \left( A\left(x^{k-1}(t)\right) x^{[k]} + B\left(x^{k-1}(t)\right) u^{[k]} \right),\ x^{[0]}(0) = x_0} \qquad (\mathbf{5.13})$$

$for\ k = 0$

$Minimize$

$$J^{[0]} = t_f \int_0^1 (x^{[0]^T} Q x^{[0]} + u^{[0]^T} R u^{[0]}) d\tau \qquad (5.14)$$

$Subject\ to$

$$\boldsymbol{\frac{dx^{[0]}}{d\tau} = t_f \left( A(x_0) x^{[0]} + B(x_0) u^{[0]} \right),\ x^{[0]}(0) = x_0} \qquad (\mathbf{5.15})$$

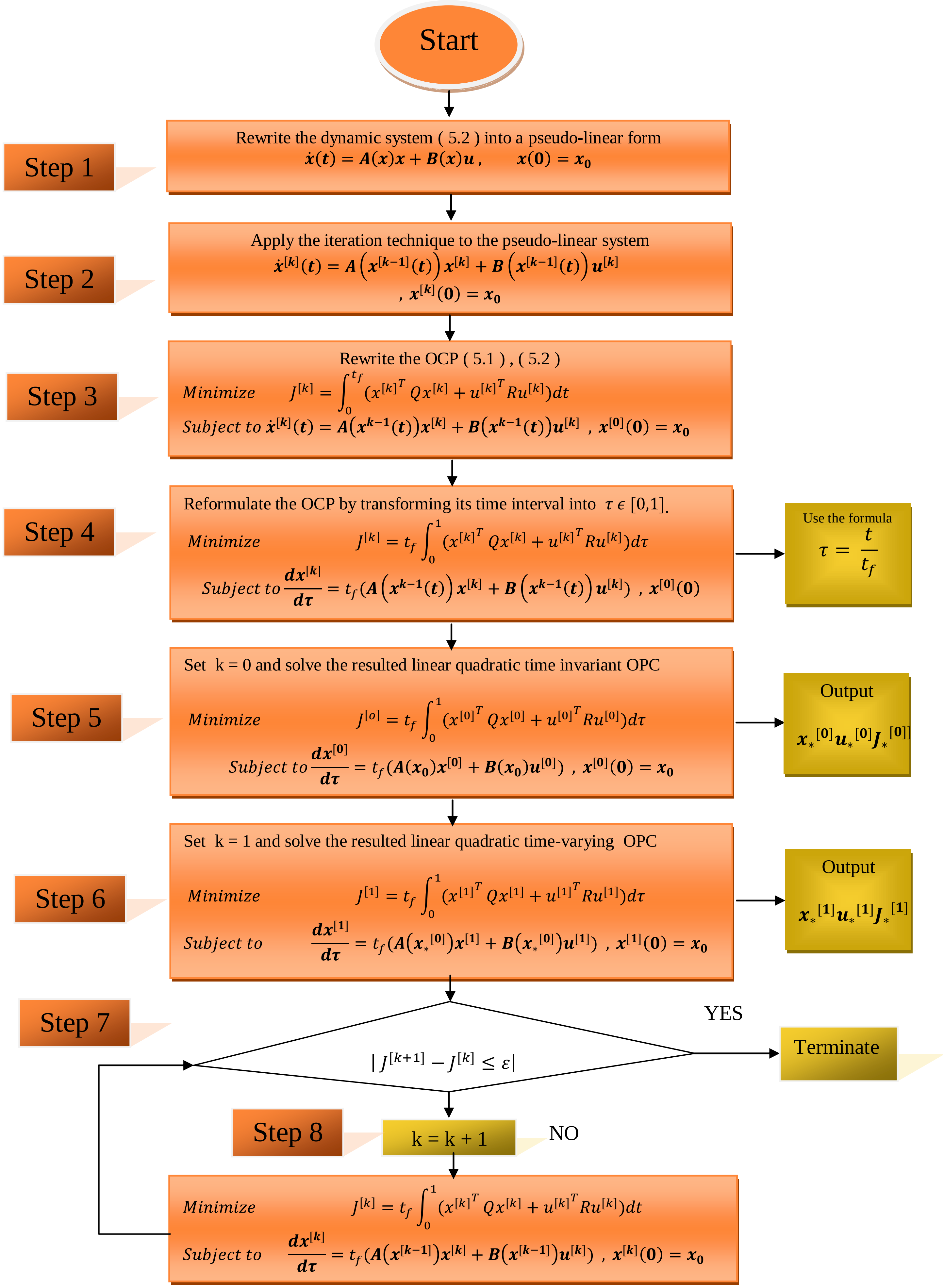


**Figure ( 5.1 )** flow chart for solving nonlinear Quadratic OCP

## 5.5. Numerical Example

**Van der Pol Oscillator problem**

Find an optimal controller $u^*(t)$ that minimizes the following performance index

$$J = \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + u^2)dt \qquad (5.16)$$

*Subject to*

$$\dot{x_1} = x_2 \quad , \quad x_1(0) = 1 \qquad (5.17)$$

$$\dot{x_2} = -x_1 + x_2 - x_1^2 x_2 + \mathrm{u} \quad , \quad x_2(0) = 0 \qquad (5.18)$$

Using the iteration technique the problem became as

Minimize

$$J^{[k]} = \frac{1}{2}\int_0^5 ((x_1^{[k]})^2 + \left(x_1^{[k]}\right)^2 + \left(u^{[k]}\right)^2)dt \qquad (5.19)$$

Subject to

$$\dot{x_1}^{[k]} = x_2^{[k]} \quad , \quad x_1^{[k]}(0) = 1 \qquad (5.20)$$

$$\dot{x_2}^{[k]} = -x_1^{[k]} + \left(1 - ((x_1^{[k-1]})^2\ \right) x_2^{[k]} + u^{[k]} \quad , \quad x_2^{[k]}(0) = 0 \qquad (5.21)$$

for $k = 0$

Minimize

$$J^{[0]} = \frac{1}{2}\int_0^5 ((x_1^{[0]})^2 + \left(x_1^{[0]}\right)^2 + \left(u^{[k]}\right)^2)dt \qquad (5.22)$$

Subject to

$$\dot{x_1}^{[0]} = x_2^{[0]} \quad , \quad x_1^{[0]}(0) = 1 \qquad (5.23)$$

$$\dot{x_2}^{[0]} = -x_1^{[0]} + u^{[0]} \quad , \quad x_2^{[0]}(0) = 0 \qquad (5.24)$$

Using Eq. ( 5.12 ) – ( 5.15 ) to reformulate the problem, then the problem became as

Minimize

$$J^{[k]} = \frac{5}{2}\int_0^1 ((x_1^{[k]})^2 + \left(x_1^{[k]}\right)^2 + \left(u^{[k]}\right)^2)d\tau \qquad (5.25)$$

Subject to

$$\dot{x_1}^{[k]} = 5x_2^{[k]} \quad , \quad x_1^{[k]}(0) = 1 \qquad (5.26)$$

$$\dot{x_2}^{[k]} = 5\left(-x_1^{[k]} + \left(1 - ((x_1^{[k-1]})^2\ \right) x_2^{[k]} + u^{[k]}\right) \quad , \quad x_2^{[k]}(0) = 0 \qquad (5.27)$$

for $k = 0$

Minimize

$$J^{[0]} = \frac{5}{2}\int_0^1 ((x_1^{[0]})^2 + \left(x_1^{[0]}\right)^2 + \left(u^{[0]}\right)^2)d\tau \qquad (5.28)$$

Subject to

$$\dot{x_1}^{[0]} = 5x_2^{[0]} \quad , \quad x_1^{[0]}(0) = 1 \qquad (5.29)$$

$$\dot{x_2}^{[0]} = 5(-x_1^{[0]} + u^{[0]}) \quad , \quad x_2^{[0]}(0) = 0 \qquad (5.30)$$

Starting from the linear quadratic time-invariant problem ( 5.28 )-( 5.30 ), which we will consider it as the starting nominal trajectories,
We used K=2, M=6 then,

**J=0.953123245019731**

then we solved the linear quadratic time-varying optimal control problems ( 5.25 )-( 5.27 ) for k=5 iterations, I used K=1, M=3.

Table ( 5.1 ) shows the results of the optimal values of the performance index J for each K by using Chebyshev wavelets.

**Table ( 5.1 )** $Values\ of\ Performance\ Index\ (J)\ for\ each\ K\ (Iteration)$

| Iteration k | $K = 1, M = 3$ |
|---|---|
| 0 | 0.95312324501973 |
| 1 | 1.46481948154029 |
| 2 | 1.44502432590876 |
| 3 | 1.43845253386883 |
| 4 | 1.43407262339513 |
| 5 | 1.43404253580764 |

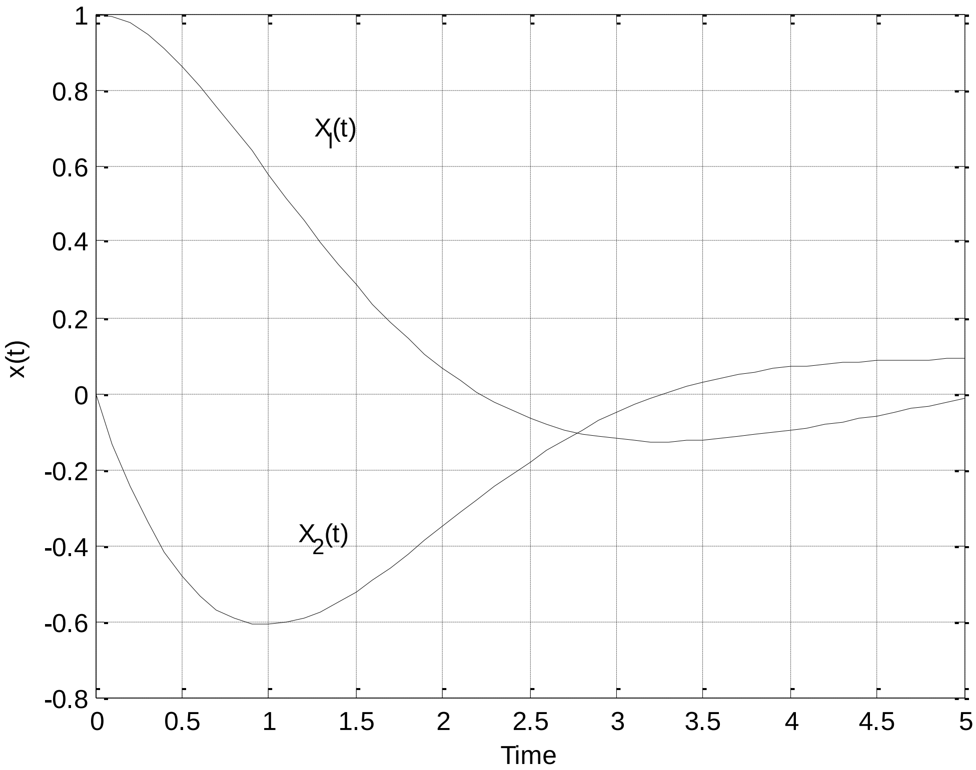


**Figure ( 5.2 )** Optimal trajectories of Van der Pol problem using Chebyshev wavelet.

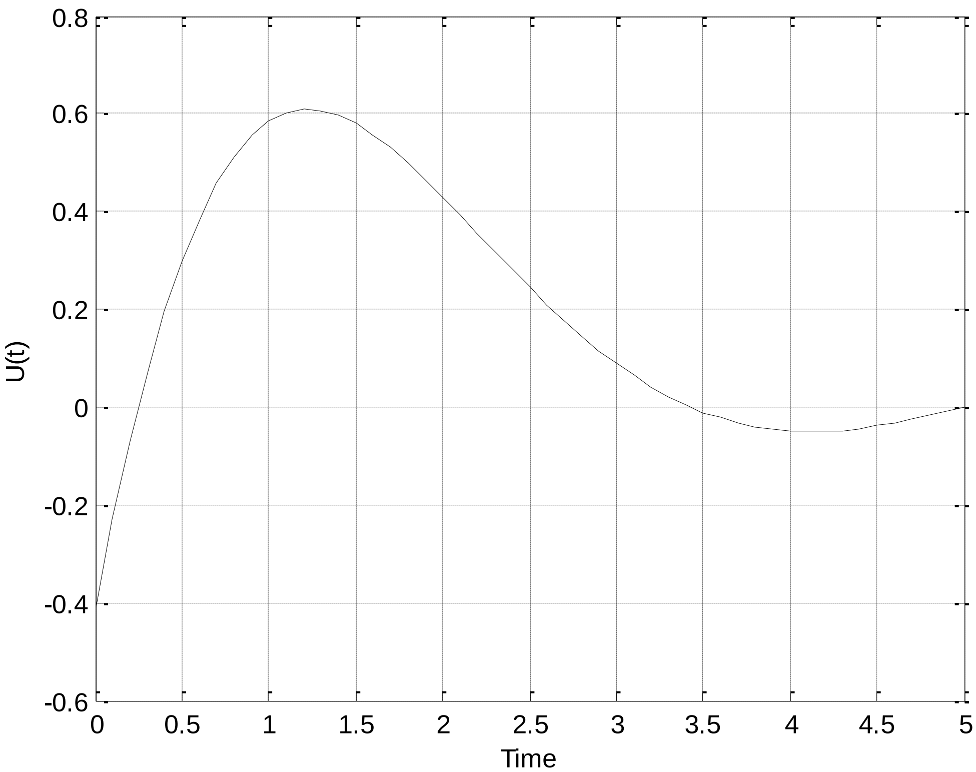


**Figure ( 5.3 )** Optimal trajectories of Van der Pol problem using Chebyshev wavelet.

Table (5.2 ) shows the comparison between different methods were solved Van der Pol oscillator problem .

**Table ( 5.2 )** Comparison between different approaches of the Van der Pol oscillator problem

| Approach Name | Performance Index ( J ) |
|---|---|
| Jaddu [2] | 1.433487 |
| Bullock and Franklin [46] | 1.433508 |
| Bashein and Enns [47] | 1.438097 |
| Majdalawi [22] | 1.4493959719 |
| This research | 1.43404253580764 |

# CHAPTER 6 CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

## 6.1. Conclusion

In this thesis, we proposed a numerical methods to solve several types of optimal control problems. We solved optimal control problem for linear time in-variant systems , linear time varying systems and nonlinear optimal systems. Theses methods are based on parameterizing the system state and control variables using a finite length Chebyshev wavelet. The aim of the proposed method is the determination of the optimal control and state vector by a direct method of solution based upon Chebyshev wavelet.

We also presented an explicit formula for the performance index. In addition Chebyshev wavelet operational matrix of integration was presented and used to approximate the solution. Also product operational matrix of Chebyshev wavelets was presented and we used it to solve linear time-varying systems.

Nonlinear optimal control problem was solved using combination between iteration technique and control-state parameterization via Chebyshev wavelets, so the solution of the difficult nonlinear optimal control problem is reduced to a simple matrix-vector multiplication solved using MATLAB program.

Compared with other methods and based on the simulation carried out in this work, our method gives better or comparable results with other methods. Using this method, the difficult linear quadratic optimal control problem is converted into a quadratic programming problem that is easy to solve.

The numerical method proposed in this thesis have many advantages as: The approximation is easy; explicit formula is presented to approximate the quadratic performance index; small quadratic programming problems are to be solved.

We applied the proposed method to several examples .The simulation results were good and the proposed method converges rapidly.

We noticed from the results of performance indices and from trajectories plots that when we increase in $M \; or \; K$ we obtained more accurate results and closed to the exact values.

## 6.2. Future Work

- In this thesis we used operational matrix of integration to approximate the solution, I suggested that we can use operational matrix of differentiation to approximate the solution.
- The work done in this thesis can be redo using Haar wavelet or Daubechies wavelet instead of Chebyshev wavelet, and solved the problem as mentioned later.

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